A SOCIAL FOUNDATION OF NASH BARGAINING SOLUTION

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Abstract. This paper provides a decentralized dynamic foundation of the Nash bargaining solution, which selects an outcome that maximizes the product of the individual gains over the disagreement outcome. We investigate a canonical search theoretic model of a society in which two agents are randomly matched, facing a pair of non-transferable payoffs drawn randomly from a compact convex set, and choose whether or not to agree to form a partnership, which is formed if and only if both of them agree to do so, subject to a small probability of exogenous break down. We show that as the discount factor converges to one, and the probability of exogenous break down vanishes, the Nash bargaining solution emerges as the unique undominated strategy equilibrium outcome. Each agent in a society, without any centralized information processing institution, behaves as if he agreed upon the Nash bargaining solution.

Keywords: Matching, Search, Undominated strategy equilibrium, Nash bargaining solution

1. Introduction

This paper studies a random matching model, in which each agent has an option to form and terminate a long term relationship with a matched partner. The society is populated with two groups of continua of agents, row and column agents. In each period, unmatched row agents and column agents are matched in pairs to face a relation-specific pair of payoffs drawn randomly from a compact convex set. They form a long term relationship if both agree upon this pair of payoffs; otherwise, they both return to their respective pools of unmatched agents. The long term relationship lasts until either one of the agents terminates it or a random shock forces the pair to separate. When the long term relationship is dissolved, both agents return to their respective pools of unmatched agents.

We demonstrate any undominated strategy equilibrium must sustain the Nash bargaining solution in the limit of the discount factor and the continuation probability of

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the partnership converging to one.[1] In the equilibrium, every agent in the society would behave as if he had agreed upon the Nash bargaining solution, despite the absence of a central authority to enforce the solution, or an institution to collect and disseminate the information in the society.[2] We thus provide a *social* foundation of the Nash bargaining solution, in the sense that the outcome arises through the interactions of all agents in the society.

The power of the axiomatic approach of Nash (1950) stems from the abstraction of details of the bargaining process. Yet, the same approach needs the strategic approach to reveal how the bargaining protocol can affect the bargaining outcome (Ståhl (1972) and Rubinstein (1982)). On the other hand, the non-cooperative foundation of the Nash bargaining solution has been subject to the criticism that the real bargaining process does not necessarily follow any pre-specified bargaining protocol, and that the proposed protocols are sensitive to seemingly minor details of the model. Raiffa (1982) eloquently describes the aspect of art in bargaining, while Kreps (1990) (pp.563-565) discusses some problems concerning non-robustness of the solution of Rubinstein (1982), with respect to a seemingly "insignificant" change in the model, including difference in the speed of response and an introduction of linear costs of rejecting offers instead of discounting. Kreps (1990) also argues that the same problem arises in any model where agents can make offers whenever they wish but have to commit to the outstanding offer for a fixed amount of time as examined by Perry and Reny (1993).

Our approach is strategic in the sense that we spell out the details of the game as a game in extensive form to invoke a refinement of Nash equilibrium. In contrast to the strategic bargaining models which spell out specific trading procedures as a part of the formal description of the model, we regard the trading procedure as a search process for an agreeable outcome between the two agents, where the outcome is characterized by the agreed payoff vector along with the probability of agreement. This search process can be interpreted as a reduced form of a complex process, involving search for a particular trading protocol and the selection of an equilibrium if multiple equilibria exist in the selected protocol.

Our model of the society is built on a canonical matching model, sharing key features with search theoretic models of markets (see, e.g., Mortensen and Pissarides (1994), Rubinstein and Wolinsky (1985) and Burdett and Wright (1998)). In contrast to assuming that the payoff of the partnership is determined by the Nash bargaining solution as in Mortensen and Pissarides (1994), we derive the Nash bargaining solution as an equilibrium outcome of the social dynamics. Instead of specifying a particular bargaining protocol as in Rubinstein and Wolinsky (1985), we delineate a class of protocols that sustain the

[1]The Nash bargaining solution is a pair of payoffs that maximizes the product of the individual gains over the disagreement outcome.

[2]Zeuthen (1930) proposed a bargaining solution more than two decades before Nash (1950). There, the risk of breakup and the amount of concession are balanced between the two parties, which is the core of the Nash bargaining solution (Nash (1953)). Zeuthen applied his concept to labor dispute, discussing the case of transferable utility only. Later, Harsanyi (1956) reformulated Zeuthen's theory to show the equivalence of Zeuthen's and Nash's concepts.

Nash bargaining solution, as a search process for an agreeable outcome to understand how robust the main conclusion of Rubinstein and Wolinsky (1985) is.[3]

Our model is essentially identical with Burdett and Wright (1998), who investigate a two-sided search model with nontransferable utility. While we focus on the limit properties of equilibrium outcomes as the discount factor and the continuation probability converge to one, Burdett and Wright (1998) are mainly concerned with economic properties of equilibrium outcomes for a given discount factor and a fixed continuation probability.[4]

The rest of the paper is organized as follows. Section 2 formally describes the basic model and investigates the properties of threshold equilibria. In section 3, we state the key result, placing the proof in the appendix. Section 4 examines the role of each assumption to understand how robust the main result is.

2. Basic Model

2.1. **Environment.** Time is discrete, $1, 2, \ldots$, and its generic element is written as t. Let $I^r = [0, 1)$ and $I^c = [1, 2)$ be the sets of continua of infinitely lived anonymous row and column agents, and their generic elements are often written as r and c, respectively. Write $I = I^r \cup I^c$. In each period, each row agent is matched with a column agent, and vice versa. There are two pools of single agents, one for row agents, and the other for column agents. The set of row (resp. column) agents in the pool in the beginning of period t is denoted by U_t^r (resp. U_t^c). Let us write $U_t = U_t^r \cup U_t^c$.

Agents in U_t^r are randomly matched with some agent in U_t^c.[5] We assume that for all $r \in U_t^r$, $c \in U_t^c$ and for all Lebesgue measurable sets $S^c \subset U_t^c$ and $S^r \subset U_t^r$,

$$\mathsf{P}(r \text{ meets someone in } S^c) = \frac{\mu(S^c)}{\mu(U_t^c)} \quad \text{and} \quad \mathsf{P}(c \text{ meets someone in } S^r) = \frac{\mu(S^r)}{\mu(U_t^r)} \tag{2.1}$$

where μ is the Lebesgue measure. As we shall see later, $\mu(U_t^r)$ and $\mu(U_t^c)$ are bounded away from zero.

The set $I \setminus U_t$ consists of the agents who agree to stay with the same partner in the previous period. Let us denote by $\mathcal{P}$ a partition of $I \setminus U$ each element of which is a pair of agents:

$$\mathcal{P} = \{\{r_\alpha, c_\alpha\}\}_\alpha.$$

If $\{i, j\} \in \mathcal{P}$, then we say that i and j are paired. Let

$$q_t = (U_t^r, U_t^c, \mathcal{P}_t)$$

[3] Young (1993) provides us with an evolutionary foundation of the Nash bargaining solution. In the sense that a random matching society is considered, our model is related to his. But again, his model assumes a specific bargaining protocol.

[4] Our model is also related to Ghosh and Ray (1996) and Fujiwara-Greve and Okuno-Fujiwara (2009), where agents are randomly matched to play a repeated game with an option to separate. Like in these models, the agents in our model form a partnership voluntarily, and can terminate the existing partnership unilaterally.

[5] Since we cannot construct a probability measure that involves "uniform" random matching (or a continuum of i.i.d. random variables in general Judd (1985)), we consider a random matching model in the spirit of Gilboa and Matsui (1992), i.e., a finitely additive measure instead of a countably additive measure, and assume that the law of large numbers holds. Note that Lebesgue measure can still be defined on I, that satisfies countable additivity.

be a coalitional structure at time t. Let Q be the set of all coalitional structures.

Suppose that $r \in U_t^r$ and $c \in U_t^c$ are matched. We assume that the two agents face a bargaining problem $\langle V, v^0 \rangle$, where V is a compact convex subset of $\mathbb{R}^2$ and $v^0 = (v_r^0, v_c^0) \in V$ is the disagreement payoff vector. In order to make the bargaining problem non-trivial, it is assumed that there exists $(v_r, v_c) \in V$ such that $v_r > v_r^0$ and $v_c > v_c^0$. We assume that the disagreement payoffs are the same across the agents of the same type.

Let us assume that a relation-specific pair of payoffs $v = (v_i, v_j) \in V$ is drawn from V according to a probability measure ν on V, where v_i is the payoff for the row agent i and v_j is the payoff for the column agent j. We assume that ν has a density function f_ν which is bounded away from zero and continuous on V. One can interpret this random process as a reduced form of a complex bargaining process, which induces v. Examples will be given subsequently.

We spell out the conditions on V and f_ν which will be used in the proof to sustain the Nash bargaining solution.

Assumption 2.1. *V is compact and convex. The measure ν has a density function f_ν such that f_ν is continuous over V, and there exists $L > 0$ such that $f_\nu(v_r, v_c) \geq L$ for all $(v_r, v_c) \in V$.*

Conditioned on one's own payoff, each agent chooses whether or not to agree to stay with the same partner in the next period: the action space of each agent is $\{A, R\}$ where A stands for "agree" and R for "reject". While we assume that the decision of agent $i \in I$ choosing A or R is conditioned only on v_i instead of (v_i, v_j), the main result remains unchanged even if we assume that each agent can observe the other party's payoff.

Given $v = (v_r, v_c)$, if $r \in U_t^r$ and $c \in U_t^c$ agree, then they form a parternship and obtain v_r and v_c, respectively. If either agent chooses R, then they remain in their respective pools of singles, receiving v_r^0 and v_c^0, respectively.

Suppose $\{r, c\} \in \mathcal{P}_t$ with the agreed payoff vector $v = (v_r, v_c)$. If both agents agree, then they remain matched for another round, i.e., $\{r, c\} \in \mathcal{P}_{t+1}$ with probability δ, receiving the same relation-specific payoffs $v = (v_r, v_c)$. But, with probability $1 - \delta$, their partnership is terminated, and they go back to their respective pools of singles, receiving v_r^0 and v_c^0, respectively. We assume that these shocks are i.i.d. across partnerships and across time. On the other hand, if either agent chooses R, then the row and column agents return to their respective pools of singles, obtain v_r^0 and v_c^0, and wait for the next period for a new match: $i \in U_{t+1}^r$ and $j \in U_{t+1}^c$.

We call $\delta < 1$ the continuation probability. Note that this assumption implies that $\mu(U_t^r) \geq 1 - \delta$ and $\mu(U_t^c) \geq 1 - \delta$ hold. The timing of matches and decisions is illustrated in Figure 1.

By a long term relationship, we mean a particular relationship that lasts for multiple periods.

Definition 2.2. *We say agents r and c are in a long term relationship at time t if there exists $k \geq 1$ such that for all $t' \in \{t - k, \ldots, t\}$,*

$$\{r, c\} \in \mathcal{P}_{t'},$$

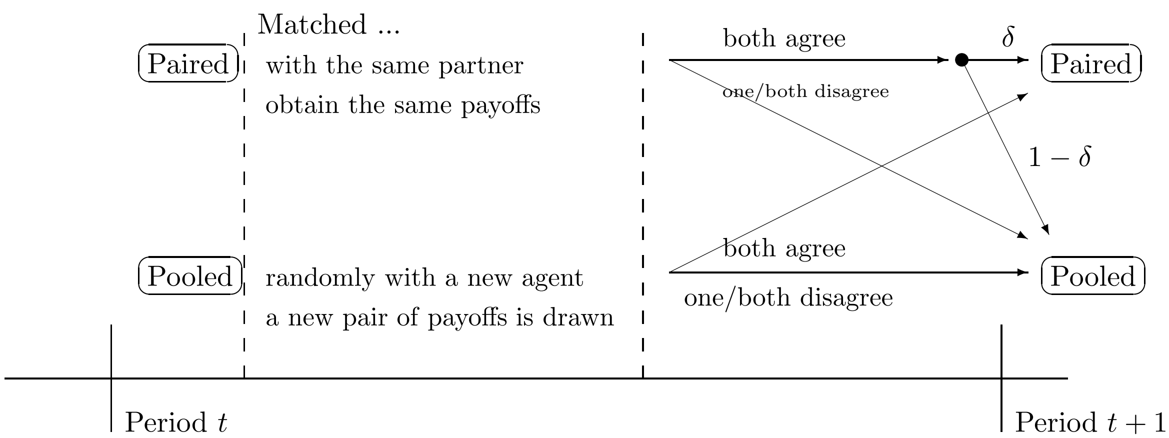

FIGURE 1. Timing of Matches and Decisions

We can interpret f_ν as a composite function of the two search processes: one for a bargaining protocol, and the other for an equilibrium outcome for a given bargaining protocol. Let us describe an example for the first step of the search process.[6]

Example 2.3. *Suppose that whenever two agents are matched, they face the divide-a-dollar game. Let V be given by*

$$V = \{(v_r, v_c) | v_r + v_c \leq 1, v_r \geq 0, v_c \geq 0\}.$$

Let $B(i, p, w)$ be the alternating offer bargaining model of Rubinstein (1982), in which agent $i \in \{r, c\}$ makes the first offer, $p \in [0, 1]$ is the continuation probability and $w \in [0, 1]$ is the size of the total surplus. One can view $1 - w$ as the amount of friction that burns part of surplus to be divided between the two agents. If the two agents do not reach an agreement before the bargaining terminates, both agents receive 0. Then the unique subgame perfect equilibrium of the bargaining game $B(r, p, w)$ is given by

$$\left(\frac{1}{1+p} w, \frac{p}{1+p} w \right),$$

and that of $B(c, p, w)$ is given by

$$\left(\frac{p}{1+p} w, \frac{1}{1+p} w \right).$$

Consider

$$\mathcal{B} = \bigcup_{i \in \{r,c\}} \bigcup_{p \in [0,1]} \bigcup_{w \in [0,1]} B(i, p, w)$$

[6] We are grateful for Michihiro Kandori for suggesting this example.

as the collection of all feasible bargaining protocols assigned to the two agents to find an agreeable outcome.

Consider a probability distribution $\tilde{\nu}$ over $\mathcal{B}$. Although $\tilde{\nu}$ is a distribution over the set of bargaining protocols, agents can calculate the equilibrium payoff vector induced by these protocols. Thus, in equilibrium, the measure $\tilde{\nu}$ induces a measure ν over V.

Suppose further that $\tilde{\nu}(i,\cdot,\cdot)$ $(i = r, c)$ has a density function over $[0,1]\times[0,1]$, and it is continuous and bounded away from zero. Then ν satisfies Assumption 2.1.

If the bargaining protocol admits multiple equilibria, we can regard ν as an equilibrium selection process used by the two agents, which is not modeled by the strategic bargaining game, as described by the next example.[7]

Example 2.4. *Suppose that the two agents agree upon bargaining protocol B, in which V is the set of equilibrium payoff vectors. To select a particular equilibrium payoff vector, the two agents hire an outsider, called the "arbitrager", who selects an outcome form $(v_r, v_c) \in V$ according to a probability measure ν, and makes a take-it-or-leave-it offer to each party. If either party rejects the offer, then both parties receive the disagreement payoff.*

We can also consider a composite of the above two examples. Conditioned on $\{r, c\}$ who are matched, a bargaining protocol $B \in \mathcal{B}$ is selected according to probability density $f^1(B)$ over $\mathcal{B}$, as in Example 2.3. Conditioned on each $B \in \mathcal{B}$, let us assume that a particular equilibrium payoff vector v of B is selected according to density function $f^2(v|B)$. Then,

$$f_\nu(v) = \int_{B\in\mathcal{B}} f^2(v|B)f^1(B)dB$$

summarizes the entire process of search for an agreeable outcome, including the negotiation over the bargaining protocol, and the negotiation within the given protocol.

2.2. **History and strategy.** We assume that each agent $i \in I$ observes only his payoff and his own action in t:

$$s_{i,t} = (v_{i,t}, r_{i,t}, d_{i,t})$$

where $v_{i,t}$ is the proposed payoff, $r_{i,t} \in \{A, R\}$ is the reaction and $d_{i,t} \in \{0,1\}$ is the coalitional status after $r_{i,t}$, 0 if $i \in U_t$ and 1 otherwise, of agent i in period t. The realized payoff $u_{i,t}$ is given by $u_{i,t} = d_{i,t}v_{i,t} + (1 - d_{i,t})v_i^0$, where $v_i^0 = v_r^0$ (resp. v_c^0) if $i \in I^r$ (resp. $i \in I^c$).

Let $h_{i,1} = \emptyset$ be the null history. At the beginning of period $t > 1$, agent i knows

$$h_{i,t} = (s_{i,1}, \ldots, s_{i,t-1})$$

which we call the private history of agent i in t. Let $H_{i,1} = \{h_{i,1}\}$, $H_{i,t}$ $(t > 1)$ be the set of all private histories of agent i in t, and $H_i = \cup_{t\geq 1} H_{i,t}$ be the set of all private histories of agent i. Let H_i be endowed with a natural measure.[8]

[7] We are grateful for Mehmet Ekmekci for the reference of Compte and Jehiel (2004).

[8] To be precise, the natural measure in this case is the product measure where the first coordinate of each $s_{i,t}$ is endowed with the Lebesgue measure, and the remaining two coordinates are endowed with the counting measures.

A strategy of agent $i \in I$ is a measurable function

$$f_i : H_i \times \mathbb{R} \to \{A, R\}.$$

Given a private history $h_{i,t}$ and a payoff $v_{i,t}$, agent i's action induced by f_i is $f_i(h_{i,t}, v_{i,t}) \in \{A, R\}$. Let $\mathcal{F}_i$ be the set of strategies of agent i. Let $h_t = (h_{i,t})_{i \in I}$ be the social history at time t. A strategy profile $f = (f_i)_{i \in I} \in \times_{i \in I} \mathcal{F}_i$ is measurable if $f^{-1}(A) = \{(h, v) | \forall i \in I \ \ [s_i(h_i, v_i) = A]\}$ is measurable. Let $\mathcal{F}$ be the set of measurable strategy profiles. Given $f \in \mathcal{F}$, let $f_{-i} \in \mathcal{F}_{-i}$ be a strategy profile of the agents except agent i where all the other agents follow f.

A strategy profile $f = (f_i)_{i \in I}$ induces a distribution over outcome paths. In period t, a social outcome is given by

$$s_t = ((s_{i,t})_{i \in I}, q_t),$$

where q_t is the coalitional structure in period t.

Given a measurable strategy profile $f \in \mathcal{F}$, the payoff function of agent i is given by

$$\mathsf{U}_i(f) = \mathsf{E}^f \left[(1-\beta) \sum_{t=1}^{\infty} \beta^{t-1} u_{i,t} \right] \tag{2.2}$$

where E^f is the expectation operator induced by f, and $\beta \in (0, 1)$ is a discount factor. We often omit superscript "f" to simply write "E".

2.3. **Solution concept.**

The basic solution concept is Nash equilibrium.

Definition 2.5. *A measurable strategy profile $f^* \in \mathcal{F}$ is a* Nash equilibrium*, or simply an* equilibrium*, if for all i, for all $f_i \in \mathcal{F}_i$,*

$$\mathsf{U}_i(f^*) \geq \mathsf{U}_i(f_i, f^*_{-i}).$$

Each agent is infinitesimal, and no public announcement mechanism exists in the society. As a result, the probability distribution over outcomes does not change even after a unilateral deviation from f^*. The continuation game off the equilibrium path is the same as the one on the equilibrium path in terms of probability distribution except that deviating to "A" will result in "punishment" from one's partner.

Given two private histories h_i, h_i', and the most recent draw v_i, define the continuation game strategy of agent i as

$$f_i(h_i', v_i | h_i) = f_i((h_i \circ h_i'), v_i)$$

where $h_i \circ h_i'$ is the concatenation of h_i and h_i'. Given history h, define $f(\cdot|h) = (f_\alpha(\cdot|h))_{\alpha \in I}$ as the profile of continuation game strategies.

Given f, let us define the continuation value of agent i following private history $h_{i,t}$ as

$$\mathsf{U}_i(f | h_{i,t}) = \mathsf{E}^f \left[(1-\beta) \sum_{k=0}^{\infty} \beta^k u_{i,t+k} \;\middle|\; h_{i,t} \right].$$

In an equilibrium, the continuation value of the agent in period t is a function of $d_{i,t-1}$ and $v_{i,t}$, where $d_{i,t-1} \in \{0, 1\}$ indicates whether the agent is in the pool of singles ($d_{i,t-1} = 0$)

or is paired ($d_{i,t-1} = 1$) in period $t-1$. Therefore, the continuation value of agent i depends only upon $d_{i,t-1}$ and $v_{i,t}$. Thus, instead of $\mathsf{U}_i(f|h_i)$, we consider

$$W_i^0(h_i) = \mathsf{U}_i(f|h_i \circ (\cdot, \cdot, 0))$$

and

$$W_i(v_i, h_i) = \mathsf{U}_i(f|h_i \circ (v_i, A, 1))$$

for any h_i and v_i where $W_i(v_i, h_i)$ is the expected continuation value conditional upon the event that the opponent accepts the proposed payoff in that period.

Since the action by a single agent does not change the distribution over social outcome, every agent in the same population faces the same distribution over states along the equilibrium path. Thus, in each period t, all agents of the same type take the same action conditioned on $d_{i,t-1}$ and $v_{i,t}$. Hence, we can represent an equilibrium by a profile of value functions,

$$(W_r^0(h_r), W_c^0(h_c); W_r(v_r, h_r), W_c(v_c, h_c))_{(v_r,v_c)\in V, h_r, h_c}$$

where r and c represent the row and column agents, respectively.

This game admits a trivial equilibrium that consists of the weakly dominated strategy

$$f_r(h_r, v_r) = f_c(h_c, v_c) = R \qquad \forall (h_r, v_r),\ (h_c, v_c).$$

To see this, note that forming a partnership requires acceptance by both parties. Thus, against the perpetual rejection by the other party, no attempt to form a partnership is successful. As a result, a pair of perpetual rejection strategies forms a Nash equilibrium. Given this pair of strategies, the continuation game payoff is (v_r^0, v_c^0). Conditioned on any (v_r, v_c) where $v_r > v_r^0$ and $v_c > v_c^0$, if an agent has a small chance of accepting the outcome, then it is the best response of the other party to accept the outcome. Thus, the perpetual rejection is a weakly dominated strategy.

The following definition of undomination, *albeit* weaker than the standard definition, suffices for our purpose. It basically states that agent i ought to accept the payoff v_i if and only if the conditional continuation value $W_i(v_i, h_i)$ of acceptance conditioned on the opponent's acceptance is greater than the continuation value $W_i^0(h_i)$ of rejection.

Definition 2.6. *An equilibrium $f^* = (f_i^*)_{i\in I}$ is an* undominated strategy equilibrium *(an* undominated equilibrium *for short) if $\forall i \in \{r, c\}$,*

$$W_i(v_i, h_i) > W_i^0(h_i) \quad \Rightarrow \quad f_i^*(h_i, v_i) = A,$$

and

$$W_i(v_i, h_i) < W_i^0(h_i) \quad \Rightarrow \quad f_i^*(h_i, v_i) = R.$$

Note that this definition does not say anything about the opponent's possibility of acceptance, and therefore, v_i may never be realized even if $W_i(v_i, h_i) > W_r^0(h_i)$ holds. For the rest of the paper, we focus on undominated equilibria.

3. Result

Let $v^N = (v_r^N, v_c^N)$ be a Nash bargaining solution:

$$v^N = \arg \max_{v \in V, (v_r, v_c) > (v_r^0, v_c^0)} (v_r - v_r^0)(v_c - v_c^0).$$

Let us now state the main result of this paper.

Theorem 3.1. *For all $\epsilon > 0$, there exist $\delta' < 1$ and $\beta' < 1$ such that if $\delta \in (\delta', 1)$ and $\beta \in (\beta', 1)$, then for all undominated equilibrium, for all $i \in \{r, c\}$, all v_i, and all h_i along the equilibrium path,*

$$|W_i^0(h_i) - v_i^N| < \epsilon,$$

and

$$|W_i(v_i, h_i) - v_i^N| < \epsilon.$$

While the main result obtains for any undominated equilibrium, it is more convenient to describe the intuition behind the theorem, using a "stationary" undominated equilibrium in which the equilibrium strategy of player i depends only upon whether player i is in the pool of singles or whether player i is in a partnership with some player j, receiving v_i as agreed upon. After we state and prove the main result for the class of stationary undominated equilibria, we prove Theorem 3.1 for all undominated equilibria.

3.1. **Stationary undominated equilibrium.**

Let us state the definition of stationary equilibrium.

Definition 3.2. *An equilibrium $f^* = (f_i^*)_{i \in I}$ is* stationary *if the distributions of coalitional structure, and one-shot actions of the agents are stationary across time: $\mu(U_t^r) = \mu(U_t^c)$ is constant across time, and for a given $v = (v_r, v_c) \in V$, $\mu(\{i \in U_t^r | \; f_i^*(h_i, v_r) = A\})$ and $\mu(\{i \in U_t^c | \; f_i^*(h_i, v_c) = A\})$ are constant across histories $(h_i)_{i \in I}$ induced by f^*.*

If a profile of value functions

$$(W_r^0(h_r), W_c^0(h_c); W_r(v_r, h_r), W_c(v_c, h_c))_{(v_r, v_c) \in V, h_r, h_c}$$

is induced by a stationary equilibrium f^*, then we can drop h_r and h_c from the argument of the value function, and write

$$(W_r^0, W_c^0; W_r(v_r), W_c(v_c))_{(v_r, v_c) \in V}.$$

We focus on the value functions of the row agent, as the computation of the column agent's value function follows the same logic. In a stationary undominated equilibrium, given v_r, agent r chooses A if

$$W_r(v_r) > W_r^0,$$

and R if

$$W_r(v_r) < W_r^0.$$

$W_r(v_r)$ is decomposed into

$$W_r(v_r) = (1 - \beta) v_r + \beta \left[\delta p_c W_r(v_r) + (1 - \delta p_c) W_r^0 \right], \tag{3.3}$$

where p_c is the probability of acceptance by the partner conditional upon the information available to agent r. From (3.3), we have

$$W_r(v_r) = \frac{1-\beta}{1-\beta\delta p_c} v_r + \frac{\beta(1-\delta p_c)}{1-\beta\delta p_c} W_r^0. \tag{3.4}$$

Observe that

$$W_r(v_r) > W_r^0$$

holds if and only if

$$v_r > W_r^0,$$

independently of the value of p_c. Hence, the optimal strategy must be a threshold rule: every row agent chooses A if $v_r > W_r^0$ and chooses R if $v_r < W_r^0$. Similarly, each column agent uses W_c^0 as an equilibrium threshold. Recall that (v_c, v_r) is drawn according to ν, which is atomless. Thus, almost surely, the equilibrium decision rule must be deterministic. Also, stationarity implies that if agent r accepts v_r in period t, then it is optimal for agent r to choose A in $t' \geq t$. In particular, if an agent chooses A to form a partnership upon drawing v_r, then it is optimal for the agent to choose A after the partnership is formed. Thus, the existing partnership is dissolved in a stationary undominated equilibrium only through the exogenous shock that arrives with probability $1-\delta$ in each period. Hence, we have $p_c = 1$.

Given the threshold decision rule, we decompose W_r^0 into

$$W_r^0 = (1-\beta)v_r^0 + \beta\left[(1-p^{W^0})W_r^0 + \int_{(v'_r,v'_c)\geq(W_r^0,W_c^0)} W_r(v')d\nu(v')\right], \tag{3.5}$$

where p^{W^0} is given by

$$p^{W^0} = \nu([W_r^0,\infty)\times[W_c^0,\infty)) = \int_{(v'_r,v'_c)\geq(W_r^0,W_c^0)} d\nu(v'), \tag{3.6}$$

the probability of the event that agent r forms a partnership with a column agent. Let $\mathcal{P}_r$ be such an event. Similarly, $\mathcal{P}_c$ is an even that a column agent c forms a partnership with a row agent r.

Solving (3.3) and (3.5) for the row agent and repeating the same calculation for the column agent, we have

$$\begin{bmatrix} W_r^0 \\ W_c^0 \end{bmatrix} = \frac{1-\beta\delta}{1-\beta\delta+\beta p^{W^0}} \begin{bmatrix} v_r^0 \\ v_c^0 \end{bmatrix} + \frac{\beta p^{W^0}}{1-\beta\delta+\beta p^{W^0}} \begin{bmatrix} \mathsf{E}(v_r|\mathcal{P}_r) \\ \mathsf{E}(v_c|\mathcal{P}_c) \end{bmatrix}, \tag{3.7}$$

or

$$(1-\beta\delta)\begin{bmatrix} v_r^0 - W_r^0 \\ v_c^0 - W_c^0 \end{bmatrix} + \beta p^{W^0} \begin{bmatrix} \mathsf{E}(v_r|\mathcal{P}_r) - W_r^0 \\ \mathsf{E}(v_c|\mathcal{P}_c) - W_c^0 \end{bmatrix} = 0. \tag{3.8}$$

We first prove the existence of a stationary undominated equilibrium.

Proposition 3.3. *A stationary undominated equilibrium exists.*

Proof. See Appendix A. □

Let (W_r^0, W_c^0) be the pair of value functions in a stationary undominated equilibrium conditioned on the event that an agent is in the pool of singles. Given (W_r^0, W_c^0), we can construct an equilibrium strategy. Given $r \in U^r$, agent r chooses A if

$$v_r > W_r^0,$$

and R if

$$v_r < W_r^0.$$

From (3.8), we know this threshold strategy is indeed an optimal strategy, since

$$v_r \geq W_r^0$$

if and only if

$$W_r(v_r) \geq W_r^0.$$

It is straightforward to prove that the described threshold rule is optimal, following every history. Since any stationary undominated equilibrium strategy is completely characterized by its threshold conditioned on the event that the agent is in the pool of singles, we can write $W^0 = (W_r^0, W_c^0)$ to represent a stationary undominated equilibrium.

Note that as $\beta\delta \to 1$, the first term in (3.8) vanishes. Therefore, the second term has to vanish, in order to satisfy the equality, which implies that $p^{W^0} \to 0$. This is the case only if $(W_r^0, W_c^0) \in V$ converges to the Pareto frontier of V as $\beta\delta \to 1$.

Lemma 3.4. *Any stationary undominated equilibrium outcome is Pareto efficient in the limit of $\beta\delta \to 1$.*

From (3.7), Lemma 3.4 implies that for $i \in \{r, c\}$,

$$\lim_{\beta\delta \to 1} |W_i^0 - \mathsf{E}(v_i|\mathcal{P}_i)| = 0,$$

which in turn implies that

$$\lim_{\beta\delta \to 1} \frac{1 - \beta\delta}{1 - \beta\delta + \beta p^{W^0}} = 0.$$

As $\beta\delta$ goes to one, the expected number of periods in which an agent stays in the pool of singles increases. However, the proportion of periods of an agent's staying in the pool converges to zero in the limit.

We now establish the conclusion of Theorem 3.1 for stationary undominated equilibria.

Proposition 3.5. *For all $\epsilon > 0$, there exist $\delta' < 1$ and $\beta' < 1$ such that if $\delta \in (\delta', 1)$ and $\beta \in (\beta', 1)$, then for all stationary undominated equilibrium, for all $i \in \{r, c\}$, all v_i,*

$$|W_i^0 - v_i^N| < \epsilon,$$

and

$$|W_i(v_i) - v_i^N| < \epsilon.$$

The proof has a geometric intuition, if V is a triangle in $\mathbb{R}^2$ with the right angle at v^0, as depicted in Figure 2. Let us first sketch the proof if ν is a uniform distribution over V, and then explain how the result can be extended to a general ν that has a continuous density function f_ν. A formal proof can be found in Appendix B.

We make a series of observations. Since $W^0 = (W_r^0, W_c^0)$ is a convex combination of vectors in V, $W^0 \in V$. If V is a right triangle with the right angle at $v^0 = (v_r^0, v_c^0)$, then the Nash bargaining solution $v^N = (v_r^N, v_c^N)$ is the middle point of the long edge (i.e., the Pareto frontier) of V. Connect v^N with v^0, and choose any point $W^0 = (W_r^0, W_c^0) \in V$. We show that if W^0 is located above the line segment connecting v^0 and v^N, (3.8) cannot hold. A symmetric argument shows that no W^0 located below this line segment can be a solution of (3.8). Thus, if W^0 solves (3.8), then it should be located on this line segment. Then, Lemma 3.4 implies that $W^0 \to v^N$ as $\beta\delta \to 1$, thus completing the proof of Proposition 3.5.

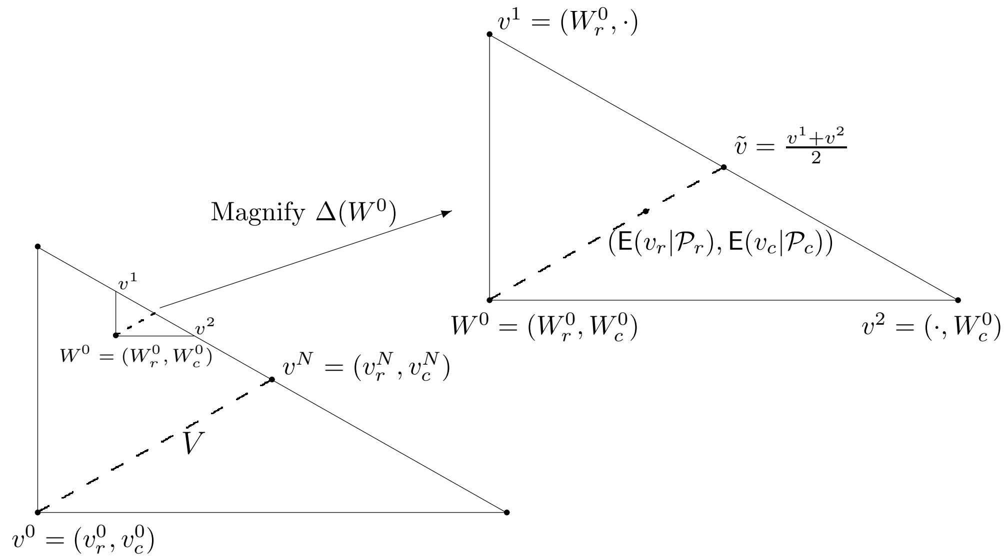

FIGURE 2. $\Delta(W^0)$ is the triangle formed by W^0, v^1 and v^2. Vector $(\mathsf{E}(v_r|\mathcal{P}_r) - W_0^r, \mathsf{E}(v_c|\mathcal{P}_c) - W_0^c)$ points to the centroid of triangle $\Delta(W^0)$, which is embedded in the line segment connecting W^0 and $\tilde{v}$. If V is a right triangle, $\Delta(W^0)$ is similar to V. The dashed line of the two triangles are parallel to each other.

Fix W^0 located above the line segment connecting v^0 and v^N. We shall show that W^0 cannot solve (3.8). Let $\Delta(W^0)$ be the collection of payoff vectors in V that Pareto dominate W^0. Since V is a right triangle, $\Delta(W^0)$ and V are similar.

Since f_ν is uniform over right triangle $\Delta(W^0)$, $(\mathsf{E}(v_r|\mathcal{P}_r), \mathsf{E}(v_c|\mathcal{P}_c))$ coincides with the centroid of $\Delta(W^0)$. Thus, vector $(\mathsf{E}(v_r|\mathcal{P}_r) - W_r^0, \mathsf{E}(v_c|\mathcal{P}_c) - W_c^0)$ is on the line segment connecting W^0 and the middle point of the long edge of right triangle $\Delta(W^0)$. Note that this line segment is parallel to the line segment in connecting v^0 and v^N.

Since $\Delta(W^0)$ and V are similar, and the long edge of $\Delta(W^0)$ is a subset of the long edge of V,

$$\frac{\mathsf{E}(v_c|\mathcal{P}_c) - W_c^0}{\mathsf{E}(v_r|\mathcal{P}_r) - W_r^0} = \frac{v_c^N - v_c^0}{v_r^N - v_r^0}.$$

Since W^0 is "above" the line segment connecting v^0 and v^N,

$$\frac{W_c^0 - v_c^0}{W_r^0 - v_r^0} > \frac{v_c^N - v_c^0}{v_r^N - v_r^0}.$$

Hence,

$$\begin{bmatrix} \mathsf{E}(v_r|\mathcal{P}_r) - W_r^0 \\ \mathsf{E}(v_c|\mathcal{P}_c) - W_c^0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_r^0 - W_r^0 \\ v_c^0 - W_c^0 \end{bmatrix} \tag{3.9}$$

are not linearly dependent, and therefore, W^0 cannot solve (3.8).

For a general distribution with a continuous density function f_ν, $(\mathsf{E}(v_r|\mathcal{P}_r)-W_r^0, \mathsf{E}(v_c|\mathcal{P}_c)-W_c^0)$ may not be equal to the centroid of $\Delta(W^0)$. The discrepancy is caused by the deviation of f_ν over $\Delta(W^0)$ from the uniform distribution. The conclusion follows if we can show that in the neighborhood of the Pareto frontier of V, f_ν over $\Delta(W^0)$ is "close" to the uniform distribution.

Note

$$\frac{\mathsf{E}(v_c|\mathcal{P}_c) - W_c^0}{\mathsf{E}(v_r|\mathcal{P}_r) - W_r^0} = \frac{\int_{v \geq W^0} (v_c - W_c^0) f_\nu(v) dv}{\int_{v \geq W^0} (v_r - W_r^0) f_\nu(v) dv}.$$

Let ρ be the distance between $W^0 \in V$ and the Pareto frontier of V. Since f_ν is continuous over a compact set V, it is uniformly continuous. Thus, $\forall \epsilon > 0$, $\exists \rho(\epsilon) > 0$ such that $\forall \rho \in (0, \rho(\epsilon))$,

$$\sup_{v \geq W^0} f_\nu(v) - \inf_{v \geq W^0} f_\nu(v) < \epsilon.$$

Define $\underline{f} = \inf_{v \geq W^0} f_\nu(v)$. Since f_ν satisfies Assumption 2.1, $\underline{f} > 0$. Then, we have

$$\left[\frac{\underline{f}}{\underline{f} + \epsilon}\right] \frac{\int_{v \geq W^0} (v_c - W_c^0) dv}{\int_{v \geq W^0} (v_r - W_r^0) dv} \leq \frac{\mathsf{E}(v_c|\mathcal{P}_c) - W_c^0}{\mathsf{E}(v_r|\mathcal{P}_r) - W_r^0} \leq \left[\frac{\underline{f} + \epsilon}{\underline{f}}\right] \frac{\int_{v \geq W^0} (v_c - W_c^0) dv}{\int_{v \geq W^0} (v_r - W_r^0) dv}.$$

As $\beta\delta \to 1$, W^0 converges to the Pareto frontier and $\rho \to 0$. Hence, we have

$$\lim_{\epsilon \to 0} \lim_{\beta\delta \to 1} \frac{\mathsf{E}(v_c|\mathcal{P}_c) - W_c^0}{\mathsf{E}(v_r|\mathcal{P}_r) - W_r^0} = \frac{v_c^N - v_c^0}{v_r^N - v_r^0}$$

as desired.

3.2. **Undominated equilibrium.** Since each agent is infinitesimal, Proposition 3.5 holds for all undominated equilibria. To see this, we need to characterize the set of the undominated equilibrium value functions, when each player observes (v_r, v_c). Abusing notation, let

$$W_i = (W_i^0, (W_i(v))_{v \in V}) \qquad \forall i \in \{r, c\}$$

be an equilibrium value function of player i conditioned on his *present* state and $v \in V$ following a history h_i. We intentionally suppress private history h_i. Since each individual agent is infinitesimal, every row agent and every column agent must use the same equilibrium strategy. Thus, we can represent an undominated equilibrium as

$$W = (W_r^0, W_c^0; (W_r(v)), (W_c(v)))_{v \in V}.$$

Let $\mathcal{W}$ be the set of all undominated equilibrium value functions. Since the structure of the game remains the same in every period, the set of all undominated equilibrium value functions remains the same over periods.

Let $\mathcal{F}$ be the collection of all bounded functions of V, endowed with the topology induced by the pointwise convergence of functions. Clearly, $\mathcal{W} \subset V \times \mathcal{F}^2$.

Lemma 3.6. *$\mathcal{W}$ is compact.*

Proof. See Appendix D $\square$

Consider a probability measure ξ over $\mathcal{W}$. We can write down a convex combination of value functions in $\mathcal{W}$ as a positive linear functional

$$L(\xi) = \int_{W \in \mathcal{W}} W d\xi(W).$$

Define the convex hull of $\mathcal{W}$ as the collection of all possible convex combination of value functions in $\mathcal{W}$. Using our notation, we can write the convex hull as

$$\mathsf{co}(\mathcal{W}) = \{L(\xi) \mid \xi \text{ is a probability distribution over } \mathcal{W}\}.$$

We call ξ degenerate, if ξ is concentrated at a particular element $W \in \mathcal{W}$. Otherwise, we call ξ non-degenerate.

We endow a metric over $\mathsf{co}(\mathcal{W})$ in terms of weak convergence of measures. Clearly, $\mathsf{co}(\mathcal{W})$ is convex and compact.

Definition 3.7. *A point $L \in \mathsf{co}(\mathcal{W})$ is an extreme point, if there is no non-degenerate ξ such that $L = L(\xi)$. Given a set X, let $\mathsf{e}(X)$ be the collection of all extreme points of X.*

We use the following result, known as Krein-Milman theorem.

Lemma 3.8. *Suppose that X is convex and compact. Then,*

$$X = \mathsf{co}(\mathsf{e}(X)).$$

Proof. See Royden (1988). $\square$

By invoking Krein-Milman theorem to (3.10),

$$\mathsf{co}(\mathcal{W}) = \mathsf{co}(\mathsf{e}(\mathsf{co}(\mathcal{W}))).$$

Since $\emptyset \neq \mathcal{W} \subset \mathsf{co}(\mathcal{W})$, Krein-Milman theorem implies

$$\mathsf{e}(\mathsf{co}(\mathcal{W})) \neq \emptyset.$$

By the definition of an extreme point, if $L \in \mathsf{e}(\mathsf{co}(\mathcal{W}))$, then we can identify L as a point in $\mathcal{W}$. Thus,

$$\emptyset \neq \mathsf{e}(\mathsf{co}(\mathcal{W})) \subset \mathcal{W}. \tag{3.10}$$

Under this convention, we regard an extreme point as a point in $\mathcal{W}$ instead of a linear functional.

Proposition 3.9. *An extreme point $W \in \mathsf{e}(\mathsf{co}(\mathcal{W})) \subset \mathcal{W}$ can be sustained by a stationary undominated equilibrium.*

Proof. See Appendix E. $\square$

Proposition 3.5 shows that any equilibrium in which the decision rule is stationary, the pair of value functions (W_r^0, W_c^0) satisfies

$$\lim_{\beta\delta\to 1}(W_r^0, W_c^0) = (v_r^N, v_c^N). \tag{3.11}$$

Thus, if $W = (W_r^0, W_c^0, (W_r(v)), (W_c(v))) \in \mathcal{W}$ is an extreme point, then it can be sustained by a stationary undominated equilibrium and (3.11) holds. Since

$$\mathcal{W} \subset \mathsf{co}(\mathcal{W}) = \mathsf{co}(\mathsf{e}(\mathsf{co}(\mathcal{W}))),$$

any pair of undominated equilibrium value functions converges to the Nash bargaining solution in the sense of (3.11), if each extreme point converges to the Nash bargaining solution.

4. Discussion

4.1. **General ν.** The main conclusion of the paper continues to hold as long as the vector $(\mathsf{E}(v_r|\mathcal{P}_r) - W_r^0, \mathsf{E}(v_c|\mathcal{P}_c) - W_c^0)$ is on the line segment connecting $W^0 = (W_r^0, W_c^0)$ and the middle point of the long edge of the triangle formed by W^0, v^1, and v^2, as depicted in the right side of Figure 2:

$$\frac{\mathsf{E}(v_c|\mathcal{P}_c) - W_c^0}{\mathsf{E}(v_r|\mathcal{P}_r) - W_r^0} = \frac{\tilde{v}^c - W_c^0}{\tilde{v}^r - W_r^0} \tag{4.12}$$

where

$$\tilde{v} = \frac{v^1 + v^2}{2}.$$

To ensure (4.12), it is not necessary that ν has a full support over V. In particular, it is not necessary that $(\mathsf{E}(v_r|\mathcal{P}_r), \mathsf{E}(v_c|\mathcal{P}_c))$ converges to the centroid of the triangle $\Delta(W^0)$ formed by W^0, v^1 and v^2, as $\beta\delta \to 1$.

We can relax Assumption 2.1. The crucial part of Assumption 2.1 is that ν assigns strictly positive but finite density around the neighborhood of the Pareto frontier of V. Let P be the Pareto frontier of V. Define $\forall\epsilon > 0$

$$\mathsf{P}^\epsilon = \{v \in V \mid \exists v' \in \mathsf{P}, \quad \|v - v'\| < \epsilon\}$$

as the ϵ neighborhood of the Pareto frontier of V, and $f_\nu(v|\mathsf{P}^\epsilon)$ be the density function of ν conditioned on P^ϵ.

Suppose that $\exists \underline{L}, \overline{L}$ so that

$$0 < \underline{L} \le \overline{L} < \infty \tag{4.13}$$

and

$$\underline{L} \le \lim_{\epsilon\to 0}\inf \inf_{v\in\mathsf{P}^\epsilon} f_\nu(v|\mathsf{P}^\epsilon) \le \lim_{\epsilon\to 0}\sup \sup_{v\in\mathsf{P}^\epsilon} f_\nu(v|\mathsf{P}^\epsilon) \le \overline{L}. \tag{4.14}$$

If ν is concentrated on the Pareto frontier but has a continuous density function f_ν conditioned on the Pareto frontier, then $(\mathsf{E}(v_r|\mathcal{P}_r), \mathsf{E}(v_c|\mathcal{P}_c))$ is located at the middle point of the line segment connecting v^1 and v^2, not coinciding with the centroid of the triangle $\Delta(W^0)$. Yet, we obtain (4.12) to prove Proposition 3.5.

It is crucial that we have a positive $\underline{L}$ and a finite $\overline{L}$, as the example in the next section shows. As long as (4.13) holds, the search process is sufficiently dispersed over the Pareto frontier. If $\underline{L} = 0$ or $\overline{L} = \infty$, then we essentially admit a search process, which excludes a priori some outcomes or concentrates at a particular outcome in the Pareto frontier.

4.2. ν **with an atom.** If the search for a pair of agreeable outcomes (v_r, v_c) is concentrated on a particular outcome with a positive probability, then we may have an equilibrium outcome different from the Nash bargaining solution. The next proposition shows that given ν which is concentrated at x^{**} on the Pareto frontier of V, we can construct a stationary undominated equilibrium whose long run average payoff converges to x^{**} in the limit as $\beta\delta \to 1$. One can follow the logic of the proof of Proposition 4.1 to show that there is ν, which is concentrated at a finite number of points $\{x^1, \ldots, x^K\}$ on the Pareto frontier of V for some $K < \infty$, such that we can construct a stationary undominated equilibrium whose long run average payoff vector is distributed over $\{x^1, \ldots, x^K\}$ according to ν.

Proposition 4.1. *Given V, suppose that we allow ν to have a mass point. Take any Pareto efficient outcome x^{**}. Then for all $\zeta > 0$, there exist a measure ν on V, $\bar{\beta} < 1$ and $\underline{\delta} < 1$ such that for all $\beta \in (\bar{\beta}, 1)$ and $\delta \in (\underline{\delta}, 1)$, there exists W^0 such that $\|W^0 - x^{**}\| < \zeta$, and W^0 is sustained as a stationary undominated equilibrium outcome.*

Proof. See Appendix C. □

Note that even in this case, the Nash bargaining solution is an undominated equilibrium outcome in the limit of β and δ converging to one. In other words, if there is a mass point on the Pareto frontier other than the Nash bargaining solution, there are multiple undominated equilibrium outcomes in the limit.

Note also that even if there is a mass point in the interior of V, this mass point cannot be a stationary undominated equilibrium outcome in the limit of $\beta\delta$ going to one. In other words, even with mass points of ν in the interior of V, the set of undominated equilibrium outcomes is contained in the set of Pareto efficient outcomes in the limit.

4.3. **The case of fixed β and δ.** If we assume a particular form of V and ν, we can sometimes compute the equilibrium outcome, taking β and δ given. This subsection demonstrates that we may have an asymmetric outcome, i.e., the outcome that is not on the line segment connecting the disagreement point and the Nash bargaining solution.

Suppose, as an example, that $v^0 = (0, 0)$, V is a simplex, i.e., it is given by

$$V = \{v = (v_r, v_c) \in \mathbb{R}^2 |\ v_r,\ v_c \geq 0,\ v_r + v_c \leq 1,\ v_r \leq 1/2\}.$$

To make the example tractable, let us assume that ν is distributed uniformly only on the (strong) Pareto frontier as discussed in the previous subsection.

Our equilibrium condition (3.8) remains the same[9], and we have the following condition:

$$(1 - \beta\delta)W^0 = \beta p^{W^0} \left[\mathsf{E}[v|\mathcal{P}] - W^0\right].$$

[9]To be precise, there are many other equilibrium thresholds for the column agent since any threshold below 1/2 is the same as 1/2 as they are not in the support, which does not happen in our original model due to the full support assumption.

It is verified that both W_r^0 and W_c^0 are less than or equal to 1/2. Given $W^0 = (W_r^0, W_c^0) \leq (1/2, 1/2)$, $\mathsf{E}[v|\mathcal{P}]$ is computed as

$$\mathsf{E}[v|\mathcal{P}] = \left(\frac{1}{4} + \frac{1}{2}W_r^0, \frac{3}{4} - \frac{1}{2}W_r^0\right),$$

and p^{W^0} is given by

$$p^{W^0} = 1 - 2W_r^0.$$

After tedious calculation, we obtain

$$W_r^0 = \frac{1}{2}\left(1 + \frac{1}{\beta} - \delta\right) - \frac{1}{2}\sqrt{\left(1 + \frac{1}{\beta} - \delta\right)^2 - 1},$$

and

$$W_c^0 = \frac{1}{2}\left(1 - \frac{1}{\beta} + \delta\right) + \frac{1}{2}\frac{1/\beta - \delta}{2 + 1/\beta - \delta}\sqrt{\left(1 + \frac{1}{\beta} - \delta\right)^2 - 1}.$$

It is verified that $W_r^0 < W_c^0$, while $\lim_{\beta\delta\to 1}(W_r^0, W_c^0) = (1/2, 1/2)$.

4.4. **n-types model with unanimity.** We can extend our analysis in a straightforward manner to a society consisting of n types of equally populated agents. Suppose that n agents, one from each type, are asked whether or not to form a long term relationship on a unanimity basis. They form the relationship if and only if all of n matched agents agree to do so; otherwise, they all return to their respective pools. Then we can invoke our main analysis to show that in the limit of β and δ converging to one, any undominated equilibrium outcome converges to the n-person Nash bargaining solution, i.e., the outcome v^N given by

$$v^N = (v_1^N, \ldots, v_n^N) = \arg \max_{v=(v_1,\ldots,v_n)\in V, v>v^0} (v_1 - v_1^0)\cdots(v_n - v_n^0).$$

4.5. **V is not convex.** While we model the selection of an agreeable outcome as a bargaining problem $\langle V, v^0\rangle$, in which V is usually assumed to be convex, our analysis applies even if V is not convex. To simplify exposition, let us assume that V has a differentiable Pareto frontier.

Definition 4.2. *(v_r, v_c) along the Pareto frontier of V is locally extremal if*

$$\frac{dv_c}{dv_r} = \frac{v_c - v_c^0}{v_r - v_r^0}$$

so that the slope of the Pareto frontier coincides with the slope of the contour of the Nash product

$$(v_r - v_r^0)(v_c - v_c^0)$$

at $(v_r, v_c) \in V$.

Our analysis shows that any locally extremal points along the Pareto frontier of V can be approximated by an undominated equilibrium.

Proposition 4.3. *Fix a locally extremal point v^* along the Pareto frontier of V. Then, as $\beta_n \to 1$ and $\delta_n \to 1$, there exists a sequence of stationary undominated equilibria with equilibrium value functions $W_n^0 = (W_{r,n}^0, W_{c,n}^0)$ such that $W_n^0 \to v^*$.*

If V is convex, then any locally extremal point must be the Nash bargaining solution. If V is not convex, however, some locally extremal point v^* can be a local minimizer of the Nash product over a small neighborhood of the Pareto frontier of v^*. However, if v^* is a local minimizer of the Nash product over the Pareto frontier of V, then one can argue that the convergence is not stable in the sense that if one deviates slightly from the proposed sequence of value functions, the perturbed sequence does not converge to the local minimizer v^*. This observation indicates that we need some sort of selection mechanism over the set of undominated stationary equilibria.

4.6. **Axioms of Nash.** Nash bargaining solution is completely characterized by four axioms: Invariance (INV), Symmetry (SYM), Pareto (PAR) and Independence of Irrelevant Alternatives (IIA). It is instructive to see that an undominated equilibrium in our model recovers each of four axioms in the limit of $\beta\delta$ going to one.[10]

INV requires that a solution of a bargaining problem $\langle V, v^0\rangle$ should not be affected by positive affine translation of the utility. In any undominated equilibrium, the decision rule for a representative row player to agree to form a partnership conditioned on (v_r, v_c) is a threshold rule: agree if $v_r > W_r^0$ and disagree if $v_r < W_r^0$. Since the equilibrium decision rule is invariant with respect to affine transformation of the utility, any undominated equilibrium outcome is also invariant to affine transformation.

SYM states that a solution should be symmetric if the bargaining problem is symmetric. If $\langle V, v^0\rangle$ is symmetric, then we know any undominated equilibrium outcome must converge to an egalitarian outcome in which both a row and a column player receives the same expected long run average payoff. PAR, which requires that a solution from a bargaining problem must be efficient, is implied by Lemma 3.4.

IIA states that if $\tilde{v}$ is a solution from a bargaining problem $\langle \tilde{V}, v^0\rangle$, and $\tilde{v} \in V \subset \tilde{V}$, then $\tilde{v}$ must be a solution of $\langle V, v^0\rangle$. Let v^* be the limit point of a sequence of undominated equilibrium outcomes as $\beta\delta$ converges to one. Given a pair of value functions (W_r^0, W_c^0) in a small neighborhood of $v^* \in \tilde{V}$, consider

$$\{(v_r, v_c) | v_r \geq W_r^0,\ v_c \geq W_c^0\}. \tag{4.15}$$

Suppose that $v^* \in V \subset \tilde{V}$. If the probability distribution over (4.15) in V is identical with the probability distribution over (4.15) in $\tilde{V}$, then the set of undominated equilibria for bargaining problem $\langle \tilde{V}, v^0\rangle$ coincides with that for $\langle V, v^0\rangle$.

For the sake of simplicity, suppose that both V and $\tilde{V}$ have smooth Pareto frontiers, as depicted in Figure 3. Since v^* is on the Pareto frontier of $\tilde{V}$, and $v^* \in V$, the Pareto frontier of V is approximately the same as that of $\tilde{V}$ in the neighborhood of v^*. Moreover, the slopes of the Pareto frontiers at v^* are the same for both V and $\tilde{V}$.

Given (W_r^0, W_c^0), the solution of (3.7) is completely determined by the probability distribution over

$$\{(v_r, v_c) \in \tilde{V} | v_r \geq W_r^0,\ v_c \geq W_c^0\}$$

[10] We are grateful for Stephen Lauermann for making this observation. See also Lauermann (2010).

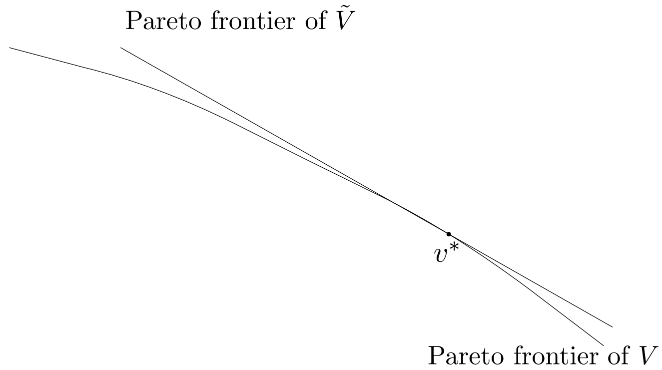

FIGURE 3. If V and $\tilde{V}$ have smooth Pareto frontiers, then the two frontiers must tangent at v^*.

and the disagreement point. Since v^* is on the Pareto frontiers of V and $\tilde{V}$, if (W_r^0, W_c^0) solves (3.7) over a small neighborhood of $v^* \in \tilde{V}$, then it must solve the same equation over a small neighborhood of $v^* \in V$. Thus, any undominated equilibrium outcome satisfies IIA in the limit of $\beta\delta$ going to one.

4.7. **Different discount factors and generalized Nash bargaining solution.** If row and column agents have different discount factors, β_r and β_c, respectively, then our result is modified so that a generalized Nash bargaining solution emerges as a unique equilibrium outcome in the limit of β_r, β_c, and δ converging to one. We can write for some $b_r, b_c, d > 0$,

$$\beta_r = e^{-\Delta b_r}, \quad \beta_c = e^{-\Delta b_c}, \quad \text{and} \quad \delta = e^{-\Delta d}$$

where $\Delta > 0$ is the time between the two rounds. We can interpret b_r and b_c as the interest rates, and d as the intensity with which the exogenous shock arrives according to a Poisson process. Define

$$\lambda = \lim_{\Delta \to 0} \frac{1 - \beta_c \delta}{1 - \beta_r \delta} = \frac{b_c + d}{b_r + d}.$$

Repeating a similar computation as we did for (3.8), we obtain

$$[1 - \beta_r\delta, 1 - \beta_c\delta] \begin{bmatrix} v_r^0 - W_r^0 \\ v_c^0 - W_c^0 \end{bmatrix} + p^{W^0} [\beta_r, \beta_c] \begin{bmatrix} \mathsf{E}(v_r|\mathcal{P}_r) - W_r^0 \\ \mathsf{E}(v_c|\mathcal{P}_c) - W_c^0 \end{bmatrix} = 0. \tag{4.16}$$

Then, we have

$$\lambda \frac{v_c^0 - W_c^0}{v_r^0 - W_r^0} = \frac{\mathsf{E}(v_c|\mathcal{P}_c) - W_c^0}{\mathsf{E}(v_r|\mathcal{P}_r) - W_r^0}$$

in the limit. Following the argument of the main analysis, we conclude that (W_r^0, W_c^0) converges to $(v_r^\lambda, v_c^\lambda)$ in the limit where we have

$$(v_r^\lambda, v_c^\lambda) = \arg \max_{v=(v_r,v_c)\in V, v>0} (v_r - v_r^0)^{\frac{1}{1+\lambda}} (v_c - v_c^0)^{\frac{\lambda}{1+\lambda}}$$

which is the generalized Nash bargaining solution.

4.8. **Two-person model.** In order to better understand the informational and institutional assumptions of our paper, let us consider the following two-person bargaining model with perfect monitoring. Two agents meet everyday until they reach an agreement. Upon agreement, they leave the market permanently. When they meet in time t, a payoff vector v in V arises according to ν. Let Assumption 2.1 hold. After observing the proposed payoff pair, they simultaneously choose to agree or disagree to v. If they both agree to v, they obtain it and leave. If not, they continue to the next period while obtaining v^0.

In this environment with perfect monitoring, we can have a result similar to the folk theorem.

Proposition 4.4. *Let* $\mathsf{int}(V)$ *be the interior of* V*. In the two-person model with perfect monitoring, take any* $v = (v_r, v_c) \in \mathsf{int}(V)$ *with* $v_r > v_r^0$ *and* $v_c > v_c^0$ *and any* $\beta < 1$*. Then there is an equilibrium in which the expected payoff vector upon agreement is* v*.*

Proof. See Appendix F. □

Since the two parties must stay together until the two parties reach an agreement, one party can punish the other party in the future after observing a deviation by the other party. We use a non-threshold strategy to construct an equilibrium with credible punishments. Moreover, the constructed perfect equilibrium induces a strict Nash equilibrium following every history so that the perfect equilibrium is an undominated equilibrium.

In our main model with a continuum of agents, each party returns to the pool of singles if at least one party rejects the proposed payoff vector. The probability that the same agents are matched in the future is 0. This feature of our model, combined with the absence of a public information aggregation process, makes it impossible to implement punishments against the other party who deviates from the equilibrium strategy.

The two-person model is sensitive to seemingly minor changes in the setup. For example, Wilson (2001) constructed a model where two agents sequentially choose to agree or not to the proposed payoff and computed a unique perfect equilibrium which is "isomorphic" to the equilibrium focused on in the present paper with $\delta = 1$.[11] On the other hand, the result in a societal model is very much independent of the choice of particular bargaining protocol, as long as the search process for an agreeable outcome is sufficiently dispersed, as formalized by Assumption 2.1.

[11] After completing the main proof, we became aware of Wilson (2001) cited in Compte and Jehiel (2004). We are grateful for Mehmet Ekmekci and Mihai Manea for the references.

Appendix A. Proof of Proposition 3.3

Choose $\overline{v} > 0$ sufficiently large so that $\forall (v_r, v_c) \in V$, $v_r < \overline{v}$ and $v_c < \overline{v}$. Clearly, $v_i^0 < \overline{v}$ $\forall i \in \{r, c\}$.

Lemma A.1. *Given W_c^0, there exists a unique W_r^0 satisfying* (3.8).

Proof. Fix W_c^0, and define

$$\varphi_r(w_r) = \frac{(1-\beta\delta)v_r^0 + \beta p^{W^0} \mathsf{E}[v_r|\mathcal{P}_r]}{1-\beta\delta+\beta p^{w^0}} - w_r.$$

The first term is a convex combination of v_r^0 and $\mathsf{E}[v_r|\mathcal{P}_r]$. In particular, if $w_r = v_r^0$, then $\mathsf{E}[v_r|\mathcal{P}_r] \geq v_r^0$. Thus,

$$\varphi_r(v_r^0) \geq 0.$$

On the other hand, if $w_r = \overline{v}$, then $p^{W^0} = 0$, since there is no $v \in V$ such that $v_r \geq \overline{v}$ and $v_c \geq W_c^0$ simultaneously. Thus,

$$\varphi_r(\overline{v}) < 0.$$

Since $\varphi_r(w_r)$ is a continuous function of w_r, $\exists W_r^0$ satisfying (3.8). A straightforward calculation shows that $\varphi(w_r)$ is a strictly increasing function at $w_r = W_r^0$ as long as $p^{W^0} > 0$. Hence, if $p^{W^0} > 0$, there can be at most one solution for (3.8). If $p^{W^0} = 0$, then $W_r^0 = v_r^0$ is the only solution for (3.8). □

Consider a function over V, which maps (w_r, w_c) to the unique solution (w_r', w_c') which solves

$$(\varphi_r(w_r'), \varphi_c(w_c')) = (0, 0).$$

Since this function is continuous over V which is convex and compact, we have a pair of (w_r, w_c) satisfying

$$(\varphi_r(w_r), \varphi_c(w_c)) = (0, 0).$$

This fixed point is the pair of value functions with the desired properties.

Appendix B. Proof of Proposition 3.5

We have shown that in any stationary undominated equilibrium, a row agent accepts v_r if $v_r > W_r^0$. Therefore, the equilibrium threshold is W_r^0, which solves the Bellman equation and therefore, satisfies

$$W_r^0 = \frac{(1-\beta\delta)v_r^0 + \beta p^{W^0} \mathsf{E}[v_r|\mathcal{P}_r]}{1-\beta\delta+\beta p^{W^0}}. \tag{B.17}$$

Similarly, the threshold for a column agent is

$$W_c^0 = \frac{(1-\beta\delta)v_c^0 + \beta p^{W^0} \mathsf{E}[v_c|\mathcal{P}_c]}{1-\beta\delta+\beta p^{W^0}}. \tag{B.18}$$

Note first that as β and δ go to one, W_r^0 and W_c^0 converge to $\mathsf{E}[v_r|\mathcal{P}_r]$ and $\mathsf{E}[v_c|\mathcal{P}_c]$, respectively. But this is impossible unless $W^0 = (W_r^0, W_c^0)$ converges to a Pareto efficient outcome.

We know

$$(1-\beta\delta)\begin{bmatrix} v_r^0 - W_r^0 \\ v_c^0 - W_c^0 \end{bmatrix} + \beta p^{W^0}\begin{bmatrix} \mathsf{E}(v_r|\mathcal{P}_r) - W_r^0 \\ \mathsf{E}(v_c|\mathcal{P}_c) - W_c^0 \end{bmatrix} = 0$$

which implies

$$\frac{v_c^0 - W_c^0}{v_r^0 - W_r^0} = \frac{\mathsf{E}(v_c|\mathcal{P}_c) - W_c^0}{\mathsf{E}(v_r|\mathcal{P}_r) - W_r^0}. \tag{B.19}$$

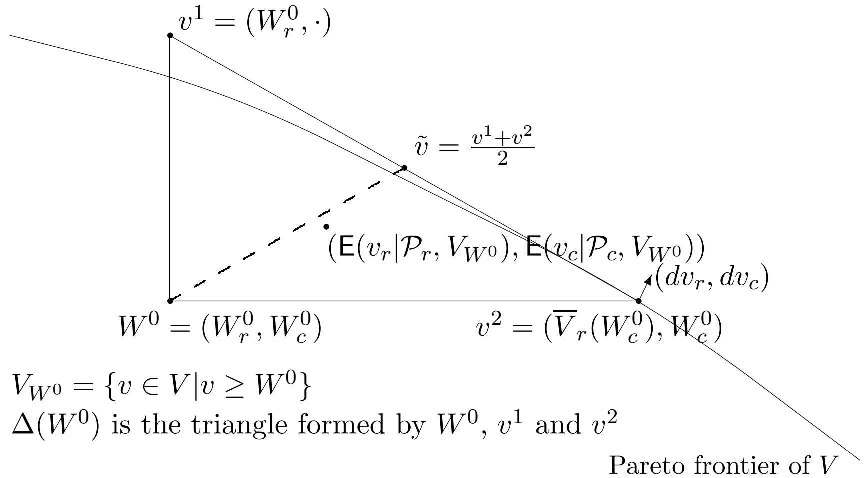

FIGURE 4. V_{W^0} is a proper subset of $\Delta(W^0)$ when the Pareto frontier is not a straight line, and W^0 is above the line segment connecting the disagreement point v^0 and the Nash bargaining solution v^N. Note that $(\mathsf{E}(v_r|\mathcal{P}_r, V_{W^0}), \mathsf{E}(v_c|\mathcal{P}_c, V_{W^0}))$ is located below the line segment connecting W^0 and $\tilde{v}$.

Given W^0, consider a right triangle $\Delta(W^0)$ with the right angle at W^0 as depicted in Figure 5. We only describe the construction of $\Delta(W^0)$ for the case where W^0 is "above" the line segment connecting v^0 and v^N. The construction for the remaining case where W^0 is "below" the same line segment follows the symmetric logic.

The following notation will be used. Given $v = (v_r, v_c)$, let

$$\overline{V}_r(v) = \sup\{v_r' | (v_r', v_c) \in V\} \quad \text{and} \quad \overline{V}_c(v) = \sup\{v_c' | (v_r, v_c') \in V\}.$$

Define

$$v^2 = (\overline{V}_r(W^0), W_c^0)$$

as the point at the Pareto frontier where the row agent can get the maximum while keeping the column agent's payoff at W_c^0. Consider the hyperplane at v^2 which separates v^2 from V. If the Pareto frontier of V is differentiable at v^2, there is a unique hyperplane with an outer norm (dv_r, dv_c) at v^2. If the Pareto frontier of V is not differentiable, the convexity of V implies that there exist more than a single separating hyperplanes separating v^2

from V. In such a case, we choose the separating hyperplane with the largest dv_c/dv_r (or the "flattest" hyperplane). Since v^2 is located on the Pareto frontier, dv_c/dv_r is strictly positive, but finite. We choose a point v^1 on the selected separating hyperplane in which the row agent's payoff is W_r^0. Let $\Delta(W^0)$ be the right triangle formed by the convex hull of W^0, v^2 and v^1.

If V is a triangle, then the long edge of $\Delta(W^0)$ is embedded in the Pareto frontier of V. In general, however,

$$V_{W^0} = \{(v_r, v_c) \in V | v_r \geq W_r^0,\ v_c \geq W_c^0\} \subset \Delta(W^0).$$

The gap between the two sets is determined by the curvature of the Pareto frontier.

Define ρ as the distance between W^0 and the Pareto frontier of V. Let us partition $\Delta(W^0)$ into two smaller triangle. Define $\tilde{v} = (\tilde{v}_r, \tilde{v}_c)$ as the middle point of v^1 and v^2:

$$\tilde{v} = \frac{v_1 + v_2}{2}.$$

Let us assume for a moment that ν is uniform, but V is convex so that V_{W^0} can be a proper subset of $\Delta(W^0)$. Suppose that $\Delta(W^0)$ is endowed with the uniform distribution. Let $f_{\nu,r}(v_r|\Delta(W^0))$ and $f_{\nu,c}(v_c|\Delta(W^0))$ be the marginal distributions of v_r and v_c conditioned on $\Delta(W^0)$, respectively. Then, consider a new pair of marginal distributions $f_{\nu,r}(v_r|V_{W^0})$ and $f_{\nu,c}(v_c|V_{W^0})$ conditioned on V_{W^0}.

Since the Pareto frontier is convex, $f_{\nu,r}(v_r|V_{W^0})$ stochastically dominates $f_{\nu,r}(v_r|\Delta(W^0))$, while $f_{\nu,c}(v_c|\Delta(W^0))$ stochastically dominates $f_{\nu,c}(v_c|V_{W^0})$. Therefore,

$$\text{(B.20)} \qquad \frac{\mathsf{E}(v_c|\mathcal{P}_c, V_{W^0}) - W_c^0}{\mathsf{E}(v_r|\mathcal{P}_r, V_{W^0}) - W_r^0} \leq \frac{\mathsf{E}(v_c|\mathcal{P}_c, \Delta(W^0)) - W_c^0}{\mathsf{E}(v_r|\mathcal{P}_r, \Delta(W^0)) - W_r^0}.$$

The desired conclusion follows from the observation that

$$\frac{\mathsf{E}(v_c|\mathcal{P}_c, \Delta(W^0)) - W_c^0}{\mathsf{E}(v_r|\mathcal{P}_r, \Delta(W^0)) - W_r^0} < \frac{W_c^0 - v_c^0}{W_r^0 - v_r^0}$$

whenever W^0 is above the line segment connecting v^0 and v^N.

From the analysis of the case where V is triangle, and f_ν satisfies Assumption 2.1, we know that if $\beta\delta < 1$ is sufficiently close to 1, we can approximate f_ν by the uniform distribution over

$$V_{W^0} = \{(v_r, v_c) | v_r \geq W_r^0, \quad v_c \geq W_c^0\}.$$

Moreover, since V is convex, (B.20) implies that (B.17) and (B.18) can solve only if W^0 is located in the small neighborhood of the line segment connecting disagreement point v^0 and the Nash bargaining solution v^N:

$$\frac{W_c^0 - v_c^0}{W_r^0 - v_r^0} = \frac{W_c^0 - v_c^N}{W_r^0 - v_r^N}.$$

Since (B.17) and (B.18) have a solution, the proof of the proposition is complete.

Appendix C. Proof of Proposition 4.1

The key idea is to construct ν in such a way that the selected outcome is sustained by a stationary undominated equilibrium. Choose an arbitrary Pareto efficient outcome $x^{**} = (x_r^{**}, x_c^{**})$ in V. Fix $\zeta > 0$. Then construct a measure ν on V in such a way that there is a mass at x^{**} with a uniform density everywhere else on V.

We would like to construct a stationary undominated equilibrium where the threshold pair W^0 is in the ζ-neighborhood of x^{**}. The equilibrium conditions are given by $\psi_r(W^0) = W_r^0$ and $\psi_c(W^0) = W_c^0$ where

$$\psi_r(W^0) = \frac{1-\beta\delta}{1-\beta\delta+\beta p^{W^0}} v_r^0 + \frac{\beta p^{W^0}}{1-\beta\delta+\beta p^{W^0}} \mathsf{E}[v_r | v \geq W^0], \tag{C.21}$$

and

$$\psi_c(W^0) = \frac{1-\beta\delta}{1-\beta\delta+\beta p^{W^0}} v_c^0 + \frac{\beta p^{W^0}}{1-\beta\delta+\beta p^{W^0}} \mathsf{E}[v_c | v \geq W^0]. \tag{C.22}$$

If $W^0 \leq x^{**}$, then (C.21) and (C.22) are rewritten as

$$\begin{aligned}\psi_r(W^0) \quad = \quad & \frac{1-\beta\delta}{1-\beta\delta+\beta p^{W^0}} v_r^0 + \frac{\beta\nu(\{x^{**}\})}{1-\beta\delta+\beta p^{W^0}} x_r^{**} \\ & + \frac{\beta\nu(\{v | v \geq W^0, v \neq x^{**}\}}{1-\beta\delta+\beta p^{W^0}} \mathsf{E}[v_r | v \geq W^0, v \neq x^{**}],\end{aligned} \tag{C.23}$$

and

$$\begin{aligned}\psi_c(W^0) \quad = \quad & \frac{1-\beta\delta}{1-\beta\delta+\beta p^{W^0}} v_c^0 + \frac{\beta\nu(\{x^{**}\})}{1-\beta\delta+\beta p^{W^0}} x_c^{**} \\ & + \frac{\beta\nu(\{v | v \geq W^0, v \neq x^{**}\}}{1-\beta\delta+\beta p^{W^0}} \mathsf{E}[v_c | v \geq W^0, v \neq x^{**}],\end{aligned} \tag{C.24}$$

respectively. Note that ψ_r and ψ_c are continuous on $V \cap (-\infty, x_r^{**}) \times (-\infty, x_c^{**})$.

If β and δ are close to one, the second term dominates the first term in both (C.23) and (C.24) so that $\psi(W^0)$ is in the ζ-neighborhood of x^{**}. Take such β and δ.

Construct a square $S_\eta = [x_r^{**} - \eta_r, x_r^{**}] \times [x_c^{**} - \eta_c, x_c^{**}]$ in the following manner. See (C.23) and (C.24) and consider $W_r^0 = x_r^{**}$. On this line, the third term becomes negligible compared to the first and second terms, while the second term still dominates the first term as W_c^0 converges to x_c^{**}. Therefore, one can find η_c such that

$$\begin{aligned}\psi_r(x_r^{**}, x_c^{**} - \eta_c) &\leq x_r^{**}, \\ \psi_c(x_r^{**}, x_c^{**} - \eta_c) &> x_c^{**} - \eta_c.\end{aligned}$$

Similarly, one can find η_r such that

$$\begin{aligned}\psi_r(x_r^{**} - \eta_r, x_c^{**}) &> x_r^{**} - \eta_r, \\ \psi_c(x_r^{**} - \eta_r, x_c^{**}) &\leq x_c^{**}.\end{aligned}$$

Using the fact that as W^0 converges to x^{**}, the third term is monotonically decreasing, while the first term is monotonically increasing, we can verify that for all x on the line segment connecting $(x_r^{**}, x_c^{**} - \eta_c)$ and x^{**}, i.e., for all $x \in \{x_r^{**}\} \times [x_c^{**} - \eta_c, x_c^{**}]$,

$$\psi(x) \leq x_r^{**}$$

holds. Similarly, it is verified that for all y on the line segment connecting $(x_r^{**} - \eta_r, x_c^{**})$ and x^{**},

$$\psi(y) \leq x_c^{**}$$

holds. Also, since the second term still dominates the first term, for all x on the line segment connecting $(x_r^{**} - \eta_r, x_c^{**} - \eta_c)$ and $(x_r^{**} - \eta_r, x_c^{**})$,

$$\psi(x) > x_r^{**} - \eta_r$$

holds, and for all y on the line segment connecting $(x_r^{**} - \eta_r, x_c^{**} - \eta_c)$ and $(x_r^{**}, x_c^{**} - \eta_c)$,

$$\psi(y) > x_c^{**} - \eta_c$$

holds. We thus verified that the boundary of S_η is mapped into S_η.

Next, we look at the interior of S_η. It is verified that $\psi_r(x_r^{**}, x_c) \leq x_r^{**}$ implies $\psi_r(x_r, x_c) \leq x_r^{**}$ for all $x_r < x_r^{**}$ since we have

$$\begin{aligned}\psi_r(x_r, x_c) = {} & \frac{1 - \beta\delta + \beta p^{(x_r^{**}, x_c)}}{1 - \beta\delta + \beta p^{(x_r, x_c)}} \psi_r(x_r^{**}, x_c) \\ & + \frac{\beta\nu(\{v | v_c \geq x_c, x_r < v_r < x_r^{**}\}}{1 - \beta\delta + \beta p^{(x_r, x_c)}} \mathsf{E}[v_r | v_c \geq x_c, x_r < v_r < x_r^{**}].\end{aligned}$$

Using a similar argument, we can verify that $\psi_c(x_r, x_c^{**}) \leq x_c^{**}$ implies $\psi_c(x_r, x_c) \leq x_c^{**}$ for all $x_c < x_c^{**}$, that $\psi_r(x_r^{**} - \eta_r, x_c) \geq x_r^{**} - \eta_r$ implies $\psi_r(x_r, x_c) \geq x_r^{**}$ for all $x_r \in (x_r^{**} - \eta_r, x_r^{**})$, and that $\psi_c(x_r, x_c^{**} - \eta_c) \geq x_c^{**} - \eta_c$ implies $\psi_c(x_r, x_c) \geq x_c^{**} - \eta_c$ for all $x_c \in (x_c^{**} - \eta_c, x_c^{**})$.

Summarizing all the above observations, it is verified that for all $W \in S_\eta$, $\psi(W) \in S_\eta$ holds. By Brouwer's fixed point theorem, there exists W^0 such that $\psi(W^0) = W^0$. This W^0 gives the equilibrium threshold pair.

Appendix D. Proof of Lemma 3.6

Since any pair of equilibrium value functions is uniformly bounded, $\mathcal{W}$ is uniformly bounded. Since the equilibrium correspondence is upper hemi-continuous, $\mathcal{W}$ is closed. Since the collection of all bounded functions over a compact set is a separable space, $\mathcal{W}$ is compact.

Appendix E. Proof of Proposition 3.9

Note that W can be written as a convex combination of today's payoff and the future value function. Since W is an extreme point of $\mathsf{co}(\mathcal{W})$, the future value function must be W. Thus,

$$W_i(v) = (1 - \beta)v_i + \beta((1 - \delta)W_i^0 + \beta W_i(v)) \qquad \forall i = r, c. \tag{E.25}$$

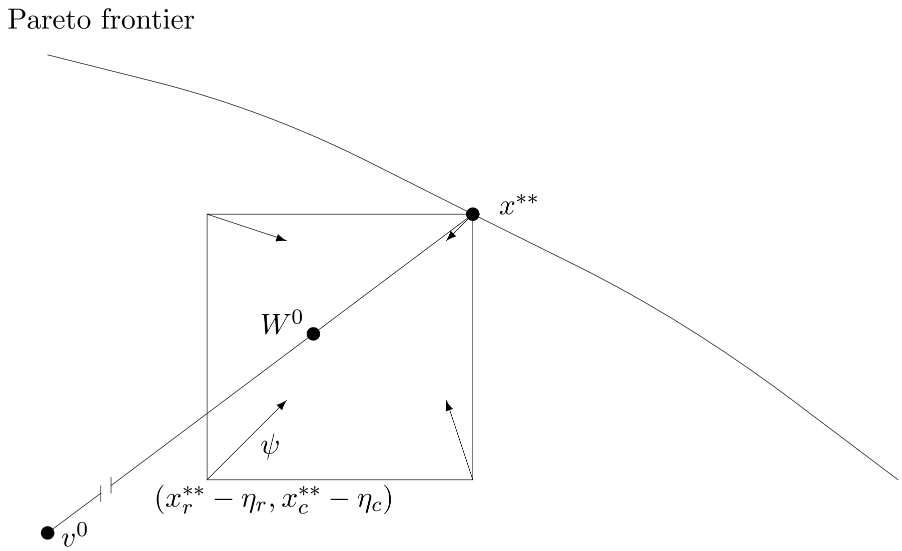

FIGURE 5. A mass on x^{**}

Note that W_i^0 and $W_i(v)$ in the right hand side are the value functions in the next period, which must be equal to the value function of today, because W is an extreme point. The optimality of the decision rule requires

$$W_i(v) > W_i^0 \ \Rightarrow A, \qquad \text{and} \qquad W_i(v) < W_i^0 \ \Rightarrow R, \qquad \forall i = r, c$$

which is equivalent to

$$v_i > W_i^0 \ \Rightarrow A, \qquad \text{and} \qquad v_i < W_i^0 \ \Rightarrow R, \qquad \forall i = r, c. \tag{E.26}$$

Since W is an extreme point, the threshold must be stationary. Moreover, each agent makes a decision, conditioned only on his own one period payoff in (v_r, v_c), even if he can observe the other agent's payoff. Given this pair of threshold rule, we have

$$W_i^0(v) = (1-\beta)v_i^0 + \beta((1-p^{W^0})W_i^0 + p^{W^0}\mathsf{E}(W_i(v)|\mathcal{P}_i)) \qquad \forall i = r, c$$

where p^{W^0} is defined as in (3.6). Since the threshold is stationary, the equilibrium strategy must be stationary, whose pair of value functions is W.

APPENDIX F. PROOF OF PROPOSITION 4.4

For the sake of simplicity, assume f_ν is constant on V and $v_r^0 = v_c^0 = 0$. Take v and β as stated in the proposition. Then take $\epsilon > 0$ so as to satisfy

$$\begin{aligned} v_r - \epsilon &> \beta v_r \\ v_c - \epsilon &> \beta v_c. \end{aligned}$$

Given $\epsilon > 0$ and $n = 0, 1, 2, \ldots$, construct a box B_n given by

$$B_n = \left[v_r - \frac{1}{n+1}\epsilon, v_r + \frac{1}{n+1}\epsilon\right] \times \left[v_c - \frac{1}{n+1}\epsilon, v_c + \frac{1}{n+1}\epsilon\right].$$

Let f_i satisfy the following:

$$f_i(h_{i,t}, v) = \begin{cases} A \text{ if there are } n \text{ deviations and } v \in B_n, \\ R \text{ otherwise.} \end{cases}.$$

This states that on the equilibrium path, they agree if and only if v is in B_0. If one agent deviates, they punish the deviation by reducing the region of acceptance from B_0 to B_1. If there is another deviation, reduce it from B_1 to B_2, and so forth. The expected payoff upon agreement is always v. And it is verified that the agents have strict incentives to follow the equilibrium strategies.

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