Time Consistency in Alternating Move Policy Games*

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Abstract

We examine alternating-move policy games where the government and the private sector alternate their moves. In contrast with the standard policy game, the set of equilibrium payoffs of the present model is bounded away from the payoff under the one-shot Nash equilibrium, called the Kydland-Prescott outcome, and the upper bound is close to the payoff under the optimal policy (called the Ramsey policy) if the government is sufficiently patient. In other words, the Kydland-Prescott outcome is not time consistent, while the Ramsey outcome could be approximated by a time-consistent policy of the same game.

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1 Introduction

In their seminal paper, Kydland and Prescott (1977) articulated the fundamental problem in implementing the optimal policy when the government does not have credible technology of commitment, and the private sector has a rational expectation about the government's policy. If the private sector responds optimally to the government's announcement of its optimal policy, then the government has an incentive to change it: the government's optimal policy may not be *time consistent*. To be time consistent, the policy must be a best response to the private sector's reaction, which, in turn, is a best response to the government's policy. This outcome, often called the Kydland-Prescott (KP) outcome, is Pareto inferior to the outcome under the optimal policy, which maximizes the objective function of the government behaving as the Stackelberg leader; this is also known as the Ramsey outcome.

The KP outcome is stable under a recursive learning dynamics, while the socially optimal outcome is not. Marcet and Sargent (1989) constructed a simple recursive algorithm in which the government learns the private sector's behavior by using the least square algorithm and demonstrated that the KP outcome is the only stable outcome of such a learning process. Even if the government initially implements the optimal policy, its policy converges to the suboptimal KP outcome as it learns more about the private sector's response. The stability result of Marcet and Sargent (1989) suggests that the policy will remain suboptimal with a high probability.

We believe that the reality of the policy is not so pessimistic as one might infer from Kydland and Prescott (1977) for three reasons. First, the history of government policy reveals that the government does not always pursue the suboptimal policy (cf. Sargent (1999)). If it has to estimate the response of the private sector recursively, then the resulting outcomes may not stay around the KP outcome. Rather, government policy exhibits a cyclical behavior that oscillates between the optimal policy and the KP outcome. As a result, the government's policy outcome Pareto dominates the KP outcome (Cho, Williams, and Sargent (2002)) in terms of the long run average social welfare.[1] Second, recent experiments by Arifovic and Sargent (1999) strongly indicate that the Ramsey outcome instead of the KP outcome appears to be the focal point of the subjects in the experiments, who move toward the Ramsey policy rather quickly. The third, and the most important, reason is that the Ramsey outcome seems far more intuitive than

[1] See also Sims (1988) and Chung (1990).

the KP outcome. If the government's objective function is to maximize social welfare, and if the Ramsey policy is a feasible solution, then it is reasonable to expect the government to make every effort to move away from the suboptimal KP outcome (Blinder (1999)).

To support our intuition and observations, it is necessary for the Ramsey policy to be time consistent so that the government's announcement of the optimal policy to be *credible*. But, it is equally important that we should have a sensible criterion to remove the suboptimal KP outcome from the set of solutions.

By modeling the problem as a repeated game between the government and the private sector, which consists of a continuum of infinitesimal agents, Chari and Kehoe (1990) and Stokey (1991) demonstrated that the Ramsey outcome can be sustained by a subgame perfect equilibrium. By the definition of subgame perfection, the government's announcement is credible following every history because the policy and the private sector's response constitute a Nash equilibrium in any continuation game. However, Chari and Kehoe (1990) and Stokey (1991) did not offer a criterion to eliminate the suboptimal KP outcome, which is also sustained by a subgame perfect equilibrium. In fact, following the idea of the folk theorem in the repeated game, we can sustain every possible payoff vector between the Ramsey outcome and the KP outcome. In the end, the pessimism of Kydland and Prescott (1977) was replaced by the multiplicity of equilibria, as Sargent (1999) put it.

In our earlier paper (Cho and Matsui (1995)), we examined a dynamic policy game in which the private sector is boundedly rational in the sense that its forecasting rule must confirm the induction principle,[2] and the government is infinitely patient. By carefully restricting the set of feasible strategies of the private sector, Cho and Matsui (1995) demonstrated that any Nash equilibrium outcome of the repeated policy game must be in a small neighborhood of the Ramsey outcome: we achieved the credibility of the Ramsey outcome while eliminating the suboptimal KP outcome. Yet, this approach, like most models with bounded rationality, requires ad hoc, albeit intuitive, restrictions on the strategies.

The present paper continues the research initiated in Cho and Matsui (1995). However, instead of boundedly rational agents investigated therein, fully rational agents are considered in this paper. In contrast with the studies

[2] Roughly speaking, the forecasting scheme of the private sector satisfies the induction principle, if after observing the same y many times, the private sector's forecast is concentrated at the small neighborhood of y.

mentioned earlier, in which the government and the private sector choose their actions simultaneously, we assume that each party chooses its action in an alternating manner as in Maskin and Tirole (1988) and Lagunoff and Matsui (1997): in every odd numbered period, the government sets its policy and in every even numbered period, the private sector chooses its response to the government's policy.[3]

By the definition of the *repeated* game, each component game, regardless of whether it is a normal form game (Stokey (1991)) or an extensive form game (Chari and Kehoe (1990)), is an *independent* game which has no physical link to the game played in the past: in every period, the players are facing the same rule of the game and the same set of choices. One way of creating linkage over time is to introduce *reputation* so that the past history can influences the strategic decision in the present round (Backus and Driffill (1985) and Barro (1986)).[4]

Instead of reputation, we introduce the linkage between the two consecutive periods by modifying the rule of the game. That is, each party can make a move not every period, but every other period. In particular, the government can make a move in every odd numbered period and the private sector moves in every even numbered period.

The alternating move game captures an important aspect of decision making processes often suppressed in the standard policy models based on the repeated games. For example, many important government policies like a tax system and private sector's actions such as investment require some time to revise. Before the government's decision is fully materialized, many private agents, if not all, have opportunities to make investment decisions. Similarly, the government makes the next move while the outcome of the private sector's investment decision has not been fully materialized.

Because of the technological constraint, the "component" game in each period is linked in such a way that the payoff of one party is influenced by the other party's action taken in the previous round. Although the alternating move game is not the most general form of payoff relevant linkage over time, it is a canonical model to investigate whether or not the KP outcome is robust against small changes of the timing of moves.

Moreover, the story of Kydland and Prescott (1977) is consistent with a sequential decision-making process between the two parties: first, the government chooses its policy, to which the private sector responds, then

[3] It is also noted that, in contrast with Cho and Matsui (1995), we assume that each party's discount factor is less than unity.

[4] See also the seminal work of Kreps, Milgrom, Roberts, and Wilson (1982) in the standard repeated game setting.

the government responds to the private sector's action, and so on. We can naturally examine the time consistency of the government's policy by examining its policy choice, which follows the response of the private sector.

The key difference of the alternating move policy game from the standard simultaneous move repeated game is that each party must (and can) commit to its action for two periods, because it can make a move only every two periods. In each period, the state variable is the decision made by the other party in the previous round. Since a decision must be carried over two periods, each party now has to consider the impact of its decision in response to the state variable (the previous action of the other party), and on the next move by the other party.

In order to illuminate the difference between the standard repeated and the alternating move policy games, we first analyze the two-by-two case that captures the essence of the policy problem of Kydland and Prescott (1977) and Stokey (1991). The difference in the set of subgame perfect equilibrium outcomes is striking. In this simple policy model, we demonstrate that the set of subgame perfect equilibrium outcomes must be bounded away from the KP outcome, and under a certain set of parameter values, the set of subgame perfect equilibrium payoffs of the government converges to the Ramsey outcome payoff. That is, the KP outcome is no longer a time consistent outcome of the alternating policy game, while all time consistent policies must induce outcomes close to the Ramsey outcome.

Since the set of strategies for the alternating move games is complex, it is difficult to completely characterize the set of subgame perfect equilibria. Instead, we focus on Markov perfect equilibrium, which requires decisions to be optimal under every "payoff relevant" state. By imposing perfection, we can inject the time consistency property in every solution of the game. At the same time, by the Markovian property, the strategy of each party depends only upon the payoff-relevant portion of a history (i.e., the opponent's action in the previous round), which significantly simplifies the analysis and enables us to completely characterize the set of equilibria.[5]

Since the private sector consists of a continuum of infinitesimal agents, the strategic power of a single agent is negligible. This feature is natural in the context of the policy game in which the government is facing a continuum of small agents. If the same alternating move game is played between two large players, then the folk theorem holds unless the interests of the two players completely coincide (Lagunoff and Matsui (1997), Yoon (2001), and Takahashi and Wen (2003)).

[5]See Maskin and Tirole (1988).

To demonstrate that the key findings from the 2×2 games can be carried over to a more complex policy game, we examine the game with a continuum of actions, focusing on Markov perfect equilibria. We choose the model of Lucas (1976) and Kydland and Prescott (1977).

We show that the set of Markov perfect equilibrium outcomes is bounded away from the KP outcome if the government is patient. If the private sector is completely impatient, we have the sharpest contrast between the simultaneous move repeated game and the alternating move game. On one hand, in Chari and Kehoe (1990) and Stokey (1991), the time preference of a representative private agent has no influence on the set of subgame perfect equilibrium outcomes. Even if one assumes that the private agent is completely impatient, one can sustain virtually all outcomes between the Ramsey and the KP outcome. On the other hand, in the alternating move game, the set of Markov perfect equilibrium outcomes converges to the optimal policy outcome as the discount factor of the government converges to unity. The Markov perfect equilibrium set not only eliminates the repetition of the KP outcome, but also converges to the Ramsey outcome.

Since the KP outcome is the Nash equilibrium of the one-shot game, it is straightforward to construct a Nash equilibrium in which the KP outcome is played repeatedly for the alternating policy game. However, this equilibrium does not satisfy the perfection requirement because the government's strategy of sticking to the KP outcome is not optimal in some unreached states. The requirement for perfection plays the same role as experiments. Instead of actually carrying out experiments to find the optimal action in every state, the perfection requirement forces the government and the private sector to deduce the optimal response under every possible state, which unravels the KP outcome.

In this setup, we characterize the bound of the steady states sustained by Markov perfect equilibria, demonstrating that the lower bound of the steady state payoffs sustained by Markov perfect equilibria is bounded away from the KP outcome, while its upper bound is close to the optimal outcome. This conclusion is consistent with our intuition that repeated interaction should offer more opportunities for the government to improve its policy, even if it does not always achieve the optimal one.

The rest of the paper is organized as follows. Section 2 defines alternating move games. Section 3 analyzes the two-by-two case which captures the essence of the policy problem of Kydland and Prescott (1977) and Stokey (1991). In this simple policy model, we demonstrate that the set of subgame perfect equilibrium outcomes must be bounded away from the KP outcome, and under a certain set of parameter values, the set of subgame perfect equilibrium outcomes converge to the Ramsey outcome payoff. We completely characterize the set of Markov perfect equilibria. Section 4 analyzes a policy model built upon Kydland and Prescott (1977) in which action spaces are closed intervals. Although a complete characterization of the Markov perfect equilibria is difficult, we show that the set of the Markov perfect equilibrium payoffs is bounded from above by the payoff of the Ramsey outcome, but the lower bound is strictly higher than the payoff of the KP outcome. Section 5 gives an example to show the structure of Markov perfect equilibria in which the unique Markov perfect equilibrium payoff is equal to the lower bound which we calculated in section 4. Combining the equilibrium outcome calculated in section 3 and section 5, we conclude that the upper bound and the lower bound calculated in section 4 are the least upper bound and the greatest lower bound, respectively. Section 6 concludes the paper.

2 Alternating Move Policy Games

2.1 The Model

We examine infinite-horizon alternating move games between the government and the private sector, which consists of a continuum of infinitesimal agents. Let $y \in Y$ be the policy variable of the government and $x \in X$ be the aggregate action of the private sector, where Y and X are the respective action spaces of the two agents. Given an action profile (y, x), let $u(y, x)$ and $v(y, x)$ be one-shot payoffs of the government and a representative agent, respectively.[6] We assume that the government's actions and the aggregate actions of the private sector are perfectly observable, but not the actions of each private agent.

Let us consider a dynamic game where the government starts the game, and moves in every odd-numbered period, while the private sector moves in every even-numbered period. Figure 1 illustrates this process. In the first period, the private sector's action is exogenously given, denoted by x_0.[7] In an odd period $2t+1$ ($t = 0, 1, 2, \dots$), if the government takes y_{2t+1} and if the private sector took x_{2t} in the previous period, then the realized payoff

[6] The payoff function v of the private sector is a reduced form of a more precise notation, $v(y, x, x')$ where the third term x' is his own action, while x is the aggregate action. We do not need this precise form in the subsequent analysis since only the representative agent's incentive will be examined. Of course, we treat the representative agent as a "price-taker", meaning that the agent does not take into account a response of the government to his own deviation.

[7] The subsequent analysis will not change even if the first action x_0 of the private sector is a choice variable instead.

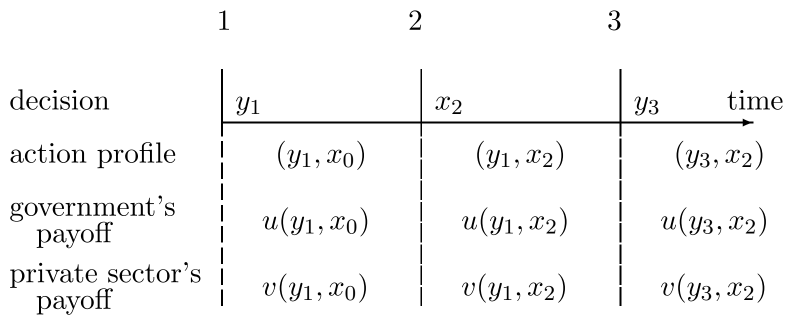

Figure 1: The play of the game

of this period to the government (resp. the private sector) is $u(y_{2t+1}, x_{2t})$ (resp. $v(y_{2t+1}, x_{2t})$). Payoffs in even periods are similarly defined.

Given a sequence of actions $(\{y_{2t-1}\}_{t=1}^{\infty}, \{x_{2t}\}_{t=1}^{\infty})$, the government's objective function at time $2T-1$ $(T = 1, 2, ...)$ is given by

$$(1-\delta_g)\sum_{t=T}^{\infty}\delta_g^{2(t-T)}\left[u(y_{2t-1}, x_{2t-2}) + \delta_g u(y_{2t-1}, x_{2t})\right], \tag{1}$$

while the private sector's objective function at time $2T$ $(T = 1, 2, ...)$ is

$$(1-\delta_p)\sum_{t=T}^{\infty}\delta_p^{2(t-T)}\left[v(y_{2t-1}, x_{2t}) + \delta_p v(y_{2t+1}, x_{2t})\right], \tag{2}$$

where $\delta_g, \delta_p \in (0,1)$ are the discount factors of the government and the private sector, respectively.

2.2 Equilibrium Concepts

We use two equilibrium concepts, subgame perfect equilibrium and Markov perfect equilibrium. A subgame perfect equilibrium is defined to be a strategy pair in which each player maximizes its own continuation value (either (1) or (2)) after any history given others' strategies. In order to reflect the fact that the private sector consists of a continuum of infinitesimal anonymous agents, the private sector's action x_{2t} at time $2t$ is determined to be the optimal response to y_{2t-1} and y_{2t+1}. More precisely, we have

$$v(y_{2t-1}, x_{2t}) + \delta_p v(y_{2t+1}, x_{2t}) \geq v(y_{2t-1}, x') + \delta_p v(y_{2t+1}, x')$$

for all x'.[8] This is because a single private agent never affects the future actions of the government.

In the sequel, we sometimes focus on a simple class of strategies where each party's action in each period is conditioned only on the payoff relevant state of the history. In the alternating move game, the state for the government is the most recent action by the private sector, and similarly, the government's most recent action is the state for the private sector. By a Markov strategy of the government, we mean a function r of the form:

$$r : X \to Y.$$

Similarly, a Markov strategy of the private sector is

$$g : Y \to X.$$

Given (r, g) and an initial state $x \in X$, the value function of the government is written as

$$\mathcal{U}(x) = \max_y \left[(1 - \delta_g)[u(y, x) + \delta_g u(y, g(y))] + \delta_g^2 \mathcal{U}(g(y)) \right]. \qquad (3)$$

Definition 2.1 *By a Markov perfect equilibrium (MPE), we mean a pair (r, g) of Markov strategies in which for all $x \in X$, $r(x)$ is a solution to (3), and for all $y \in Y$, $g(y)$ satisfies*

$$g(y) \in \max_x \left[v(y, x) + \delta_p v(r(g(y)), x) \right].$$

We shall admit mixed strategies for the both players. To simplify notation, however, we shall use r and g to represent a mixed strategy of the government and the private sector, respectively, whenever the meaning is clear from the context. With mixed strategies, (3) becomes

$$\begin{aligned} E\mathcal{U}(x) &= \max_{\hat{r}} \sum_y \hat{r}(x)(y) \\ &\quad \times \left[(1 - \delta_g) u(y, x) + (1 - \delta_g)\delta_g \sum_{x'} g(y)(x')[u(y, x') + E\mathcal{U}(x')] \right] \end{aligned}$$

where $\hat{r}(x)(y)$ is the probability (density) assigned to y by a mixed strategy $\hat{r}(x)$, and similarly, $g(y)(x)$ is the probability (density) assigned to x by $g(y)$. In this specification, a mixed strategy g of the private sector is interpreted as the situation in which agents' actions are correlated, using, say, sunspots.

[8] To be precise, this equilibrium is different from the standard definition of subgame perfect equilibrium, modified so as to reflect the infinitesimality of the private agents.

3 2×2 Policy Game

This section analyzes alternating move games with two actions for each player. The games examined in this section essentially capture the nature of the policy game of Kydland and Prescott (1977) and Stokey (1991). The stage payoff is given by (4) where $\mu < 1$.

$$\begin{array}{ccc} & & \textbf{Private} \\ & & \begin{array}{cc} L & R \end{array} \\ \textbf{Gov}. & \begin{array}{c} C \\ D \end{array} & \begin{array}{|c|c|} \hline 3,\ 3 & 0,\ 0 \\ \hline 4,\ \mu & 1,\ 1 \\ \hline \end{array} \end{array} \tag{4}$$

Note that (C, L) corresponds to the Ramsey outcome, the best outcome if the government can commit to its action, and (D, R) corresponds to the Kydland-Prescott outcome, a one-shot Nash equilibrium if we regard (4) as a one-shot game.

Let us now look at subgame perfect equilibria. Suppose $\mu > 1 - 3/\delta_p$. Since we have

$$v(C, L) + \delta_p v(D, L) = 3 + \delta_p \mu > \delta_p = v(C, R) + \delta_p v(D, R),$$

$y_{2t-1} = C$ implies $x_{2t} = L$. Therefore, even in the worst case, the government can guarantee itself the payoff of 3 from the next period on by taking C forever after. This leads to the following result, which is stated without a proof.

Theorem 3.1 *Suppose $\mu > -2$. For any $\varepsilon > 0$, there exists $\bar{\delta}_g < 1$ such that $\delta_g > \bar{\delta}_g$ implies that in any subgame perfect equilibrium, after any history, the continuation value of the government is greater than $3 - \varepsilon$.*

An important point is that the set of equilibrium outcomes is bounded away from the KP outcome, and every outcome in the set Pareto dominates the KP outcome. It should be noted that there is a Nash equilibrium of the alternating move game in which the government always plays D and the private sector takes R following every history. By subgame perfection, however, if the government's policy is C, then the private sector must choose L instead of R: the private sector's action in state C is not subgame perfect. If the government knows that the private sector will choose L in response to C, it has an incentive to deviate to C from D. The repetition of the KP outcome is not sustained by a subgame perfect equilibrium: it is not time consistent in the alternating move policy game, while all time consistent outcomes generate the long run average payoff close to the Ramsey outcome.

Next, we focus on Markov perfect equilibria to completely characterize the set of equilibrium outcomes. A Markov strategy of the government can be expressed by a pair of mixed actions $((1-p_L)[C]+p_L[D],(1-p_R)[C]+p_R[D])$, or simply (p_L, p_R), where p_L (resp. p_R) is the probability of the government's taking D when the private sector took L (resp. R) in the previous period. Similarly, (q_C, q_D) is a Markov strategy of the private sector where q_C (resp. q_D) is the probability of taking R if the government's previous action is C (resp. D).

There are eight states in this process, four pairs of actions with the indication of the mover of the period. For example, by CL we mean the state in which the government took C in the previous period, and the private sector takes L in the present period. Let V_{xy} be the discounted average payoff of the government when it takes $y \in \{C, D\}$, while the opponent's previous action is $x \in \{L, R\}$. On the other hand, we let V_{yx} be the discounted average payoff to the government when the private sector takes $x \in \{L, R\}$, while the government's previous action is $y \in \{C, D\}$.

Since Markov perfection requires that each party's action be optimal under every state, its state contingent choice can be represented as a solution of the Bellman equation. Then, we have a rather long list of recursive representations for a Markov strategy profile $((p_L, p_R),(q_C, q_D))$. In the remaining part of this section, we assume that the government and the private sector have a common discount factor, i.e., $\delta_g = \delta_p = \delta$.

$$
\begin{aligned}
V_{LC} &= (1-\delta)3 + \delta\left[(1-q_C)V_{CL} + q_C V_{CR}\right], \\
V_{LD} &= (1-\delta)4 + \delta\left[(1-q_D)V_{DL} + q_D V_{DR}\right], \\
V_{RC} &= (1-\delta)\cdot 0 + \delta\left[(1-q_C)V_{CL} + q_C V_{CR}\right], \\
V_{RD} &= (1-\delta) + \delta\left[(1-q_D)V_{DL} + q_D V_{DR}\right].
\end{aligned}
$$

And similarly,

$$
\begin{aligned}
V_{CL} &= (1-\delta)3 + \delta\left[(1-p_L)V_{LC} + p_L V_{LD}\right], \\
V_{CR} &= (1-\delta)\cdot 0 + \delta\left[(1-p_R)V_{RC} + p_R V_{RD}\right], \\
V_{DL} &= (1-\delta)4 + \delta\left[(1-p_L)V_{LC} + p_L V_{LD}\right], \\
V_{DR} &= (1-\delta) + \delta\left[(1-p_R)V_{RC} + p_R V_{RD}\right].
\end{aligned}
$$

For the private sector, we have different expressions for the value functions. Each agent is infinitesimal and, therefore, does not unilaterally affect the future outcome path except that the agent himself has to commit to an action for two periods. For this reason, we have the following eight expressions.

$$
\begin{aligned}
\Pi_{CL} &= 3+\delta\left[(1-p_L)3+p_L\mu\right], \\
\Pi_{DL} &= \mu+\delta\left[(1-p_L)3+p_L\mu\right], \\
\Pi_{CR} &= 0+\delta\left[(1-p_R)\cdot 0+p_R\cdot 1\right] \\
&= \delta p_R, \\
\Pi_{DR} &= 1+\delta\left[(1-p_R)\cdot 0+p_R\cdot 1\right] \\
&= 1+\delta p_R.
\end{aligned}
$$

and

$$
\begin{aligned}
\Pi_{CL\backslash R} &= 0+\delta\left[(1-p_L)\cdot 0+p_L\cdot 1\right] \\
&= \delta p_L, \\
\Pi_{DL\backslash R} &= 1+\delta\left[(1-p_L)\cdot 0+p_L\cdot 1\right] \\
&= 1+\delta p_L, \\
\Pi_{CR\backslash L} &= 3+\delta\left[(1-p_R)3+p_R\mu\right], \\
\Pi_{DR\backslash L} &= \mu+\delta\left[(1-p_R)3+p_R\mu\right].
\end{aligned}
$$

The first four expressions are the equilibrium two-period payoffs. For example, Π_{CL} is the two-period payoff if the government's previous action is C, and the prescribed action is L, which the agent in question follows. On the other hand, the second four expressions are the two-period payoffs when the agent makes a unilateral deviation. For example, $\Pi_{CL\backslash R}$ is the payoff of a single private agent playing R while the government's previous action is C and all other agents choose L.

The following proposition completely characterizes the set of Markov perfect equilibria for the discount factor greater than 0.5.

Proposition 3.2 *Let $\delta > 1/2$. Then the set of MPE's is characterized as follows.*

1. *if $\mu \geq 1-3\delta$, then $((p_L, p_R), ([L], \frac{1}{3\delta}))$ where*

$$p_L \leq \frac{\mu+3\delta-1}{\delta(4-\mu)},$$

and

$$p_R \geq \frac{\mu+3\delta-1}{\delta(4-\mu)}.$$

2. *if* $1-3\delta \geq \mu > 1-3/\delta$, *then* $(([C],[C]),([L],[R]))$

3. *if* $\mu < 1-3/\delta$, *then* $(([C],[C]),([L],[R]))$, $(([D],[D]),([R],[R]))$ *and* $((p_L,p_R),(1-\frac{1}{3\delta},[R]))$ *where*

$$p_L \leq \frac{3(1+\delta)}{\delta(4-\mu)},$$

and

$$p_R \geq \frac{3(1+\delta)}{\delta(4-\mu)}.$$

Proof. See Appendix A.

The first type of equilibrium arises when μ is not too small, or greater than $1-3/\delta$, e.g., $\mu = 0$. In every subgame perfect equilibrium of such games, the expected payoff of the government is 3 at state L, which is the payoff from the Ramsey outcome of the one shot policy game. Note that in this equilibrium, the government sometimes deviates from the Ramsey outcome to attain a higher payoff, and the private sector reacts to it probabilistically. Once the system falls into the Kydland-Prescott outcome, it reverts back to the Ramsey outcome by the government's shift in policy. In some sense, this forms a cycle, in which the government's policy oscillates between the KP outcome and the Ramsey outcome.[9]

As δ goes to one, the range of μ satisfying Case 2 vanishes. In the limit, the threshold between Case 1 and Case 3 becomes $\mu = -2$. This threshold is important since L becomes the unique best response whenever C is taken in the previous period if (C,L) risk-dominates (D,R), and R becomes the unique best response whenever D is taken in the other case. Indeed, in Case 3, the government's expected payoff is 1 at any state in the limit.

4 Policy Games with a Continuum of Actions

While we obtain a complete characterization of the Markov perfect equilibria in a simple game, agents in a typical policy game are endowed with a continuum of feasible actions rather than just two. In order to show the key findings in 2×2 games can be extended, this section analyzes the alternating

[9] While this observation is somewhat consistent with the findings of Sargent (1999), it is largely due to the fact that the action space of the government is discrete, as the later analysis reveals.

games in which the action space of each player is a closed interval $[0, \overline{\alpha}]$ on the real line ($\overline{\alpha} > 0$).

We would like to examine a situation examined in Lucas (1976) and Kydland and Prescott (1977) in which the government controls the inflation rate y and the private sector's "choice" variable x is interpreted as the anticipated inflation rate. Following Sargent (1999), we let

$$v(y, x) = -(y - x)^2 \tag{5}$$

so that the private sector's objective is to correctly forecast the inflation rate selected by the government.

As for the government's objective function, Sargent (1999) assumes

$$u(y, x) = -\frac{1}{2}\left[y^2 + (U^* - \theta(y - x))^2\right], \tag{6}$$

where U^* is the natural unemployment rate, and $U = U^* - \theta(y - x)$ with $\theta > 0$ represents the Phillips curve, which represents the relationship between the inflation rate y and unemployment U.

While the quadratic utility function u simplifies the calculation, we need only the following properties of u. Let $B(x)$ be the one-shot best response of the government against x.

A1. $u(y, x)$ is twice continuously differentiable and strictly quasi-concave, and $B(x)$ is an increasing function of x. For a given y, $u(y, x)$ is a strictly decreasing function of x.

A2. $\forall x < x'$, $\forall y < y' \leq B(x)$

$$0 < u(y', x) - u(y, x) < u(y', x') - u(y, x').$$

A3. The component game has a pure strategy Nash equilibrium (y_N, y_N) and a pure strategy Stackelberg equilibrium where the government is a leader. The Stackelberg equilibrium strategy y_R of the government, called the *Ramsey policy*, is normalized to 0 so that

$$0 = y_R < y_N \leq \overline{\alpha}.$$

Let

$$\varphi(\alpha) = \left.\frac{dy}{dx}\right|_{u=u(\alpha,\alpha)} = -\left.\frac{\partial u/\partial x}{\partial u/\partial y}\right|_{x=y=\alpha},$$

for $\alpha \in [0, \overline{\alpha}]$. This is the slope of the indifference curve at (α, α): the marginal rate of substitution between y and x at (α, α). This function plays an important role in characterizing the range of the Markov perfect equilibria.

If x is the anticipated inflation rate, one can interpret x as the level of reputation of the government. A1 says that the government's payoff is decreasing as its reputation for maintaining low inflation deteriorates: the optimally forecasted inflation rate is monotonically related to the actual inflation rate. Recall that the government is trying to exploit the trade-off between unemployment and inflation, which induces the government to use high inflation policy. By the same token, if the government wants to restore its reputation as an inflation fighter, it must implement low inflation policy, which may decrease its one-shot payoff. A2 says that the cost of restoring reputation increases as the government's reputation deteriorates. A3 preserves some important features of the policy game which uses (6). It is straightforward to verify that at the Ramsey outcome,

$$\varphi(y_R) \geq 1,$$

and at the component game Nash equilibrium,

$$\varphi(y_N) = \infty.$$

Although we could generalize v, too, we keep it in its current form to simplify our exposition.

In an equilibrium, given the government's policy y, the private sector has perfect foresight about the government's future move. Thus, its response x must satisfy

$$x = \frac{y + \delta_p r(x)}{1 + \delta_p} \tag{7}$$

or equivalently, for each y, there exists a unique x so that

$$y = (1 + \delta_p)x - \delta_p r(x). \tag{8}$$

Let $F(x)$ denote the right hand side of (8):

$$F(x) \equiv (1 + \delta_p)x - \delta_p r(x).$$

Definition 4.1 *g is consistent with r if*

$$g(y) \in F^{-1}(y) \quad \forall y.$$

Definition 4.2 *r is a best response to g if for $\forall x$, $r(x)$ solves (3).*

The next proposition follows from the definition.

Proposition 4.3 *(r, g) is a Markov perfect equilibrium if and only if r is a best response to g, and g is consistent with r.*

There may exist a Markov perfect equilibrium, which has no steady state, or whose equilibrium path is chaotic. Since our main objective is to see whether or not the Kydland-Prescott outcome can be sustained as a steady state of an MPE, we shall focus on the characterization of an MPE with a steady state.

Definition 4.4 *α is a steady state if $\alpha = g(\alpha) = r(\alpha)$.*

By the definition of steady state α,

$$\alpha = g(r(\alpha)) = r(g(\alpha)).$$

Hence, if the government chooses α, then the private sector's response is α, to which α is the government's (long-run) best response. As a result, the government's policy remains α for the rest of the game.

Lemma 4.5 *For any g, if r is a best response to g, then $r(x)$ is a weakly increasing function of x.*

Proof. See Appendix B

In general, a Markov perfect equilibrium entails randomization along and off the equilibrium path. As we focus on Markov perfect equilibria with a stead state, we essentially exclude those which entail randomization along the equilibrium path. Since our analysis is based upon a subset of Markov perfect equilibria, we need to establish the existence of a Markov perfect equilibrium with steady state α.

Proposition 4.6 *There exists a Markov perfect equilibrium with a steady state.*

Proof. See Appendix C

A Markov perfect equilibrium may have more than one steady states. Yet, as the government becomes more patient, the set of steady states of a Markov perfect equilibrium converges to a single point.

Proposition 4.7 *For a fixed $\delta_p \in (0,1)$,*

$$\lim_{\delta_g \to 1} \sup \left\{\alpha' : g(\alpha') = \alpha'\right\} - \inf \left\{\alpha' : g(\alpha') = \alpha'\right\} = 0 \tag{9}$$

Proof. See Appendix D

The next proposition gives upper and lower bounds for steady states in Markov perfect equilibria.

Proposition 4.8 *Let α be a steady state of a Markov perfect equilibrium. Then,*

$$\frac{1}{\delta_g} \leq \varphi(\alpha) \leq \frac{1+\delta_p}{\delta_g}.$$

Proof. Since $r \in S$ is increasing, r is differentiable almost everywhere. By the same token, F is differentiable almost everywhere, and so is g. If r and F are differentiable at x, then the first order condition must hold with respect to $y = F(x)$:

$$\begin{aligned} &(1-\delta_g)\left[\frac{\partial u(r(x),x)}{\partial y} + \delta_g \frac{\partial u(r(x), g(r(x)))}{\partial y}\right] \\ + \quad &\left[(1-\delta_g)\delta_g \frac{\partial u(r(x), g(r(x)))}{\partial x} + \delta_g^2 \mathcal{U}'(g(r(x)))\right] g'(r(x)) = 0. \end{aligned} \tag{10}$$

By the envelope theorem,

$$\mathcal{U}'(g(r(x))) = (1-\delta_g)\frac{\partial u(r(g(r(x))), g(r(x)))}{\partial x}.$$

Then, we have

$$g'(r(x)) = -\frac{[\partial u(r(x),x)/\partial y + \delta_g \partial u(r(x), g(r(x)))/\partial y]}{\delta_g \left[\partial u(r(x), g(r(x)))/\partial x + \delta_g \partial u(r(g(r(x))), g(r(x)))/\partial x\right]}. \tag{11}$$

From (11), we have

$$g'(\alpha) = -\frac{1}{\delta_g}\frac{\partial u(\alpha,\alpha)/\partial y}{\partial u(\alpha,\alpha)/\partial x} = \frac{1}{\delta_g} \left.\frac{dx}{dy}\right|_{u=u(\alpha,\alpha)}.$$

Notice that the second term is the inverse of the marginal rate of substitution of u at (α,α). It would be more convenient to write the above equation into

$$\frac{1}{g'(\alpha)} = \delta_g \left.\frac{dy}{dx}\right|_{u=u(\alpha,\alpha)}.$$

From (8), we have

$$\frac{1}{g'(\alpha)} = 1 + \delta_p - \delta_p r'(\alpha).$$

Thus, we have

$$\frac{1}{g'(\alpha)} = 1 + \delta_p - \delta_p r'(\alpha) = \delta_g \left.\frac{dy}{dx}\right|_{u=u(\alpha,\alpha)} \tag{12}$$

Let us assume $r(0) \neq 0$. Since r is weakly increasing, it first crosses the 45 degree line ($y = x$) from above, or to be precise,

$$0 \leq \frac{d^- r(\alpha)}{dx} \leq 1,$$

where $d^- r(\cdot)/dx$ is the left derivative of r. Therefore, we have

$$1 \leq \frac{1}{g'(\alpha)} = 1 + \delta_p - \delta_p \frac{d^- r(\alpha)}{dx} = \delta_g \left.\frac{dy}{dx}\right|_{u=u(\alpha,\alpha)} = \delta_g \varphi(\alpha) \leq 1 + \delta_p. \tag{13}$$

Hence,

$$\varphi^{-1}\left(\frac{1}{\delta_g}\right) \leq \alpha \leq \varphi^{-1}\left(\frac{1+\delta_p}{\delta_g}\right).$$

Q.E.D.

Note that as $\delta_g \to 1$ and $\delta_p \to 0$, $\alpha \to 0$: if the private sector becomes very impatient while the government is patient, it can (almost) implement the Ramsey policy. Similarly, as $\delta_g \to 0$, α converges to the one-shot, Nash-equilibrium outcome: if the government becomes impatient, the only outcome is to repeat the one-shot Nash equilibrium. Note that at the Kydland-Prescott outcome, or the one-shot Nash equilibrium, the marginal rate of substitution is infinite. Thus, no MPE with a steady state can sustain the Kydland-Prescott outcome.

5 Example

A typical Markov perfect equilibrium entails a mixed strategy conditioned on a certain state, which complicates the calculation of an equilibrium for a general utility function. However, we can explicitly calculate the Markov perfect equilibrium from (10) if the government's (one-shot) payoff function is quasi-linear: a concave function of y, and a linear function of x.[10]

[10] Barro (1986) used a similar formulation to calculate an equilibrium.

While (10) offers an important clue about the equilibrium strategy of each party, it is not sufficient to nail down a particular equilibrium, because we need to check the restriction imposed by Lemma 4.5. Because the construction reveals how (10) and Lemma 4.5 must be reconciled, this example, albeit special, warrants a careful analysis.

Let

$$u(x, y) = -\frac{1}{2}y^2 + \theta U^* y - \theta U^* x. \tag{14}$$

Note that this is equivalent to (6) in the limit of θ going to zero with θU^* kept constant. Substituting (14) in (11), we obtain

$$g'(r(x)) = \frac{1}{\delta_g}\left[1 - \frac{r(x)}{\theta U^*}\right]. \tag{15}$$

Letting $y = r(x)$, we restrict y to the range of r. To calculate an equilibrium, however, we first solve this ordinary differential equation without considering the range to which it is applicable. Then, we proceed to take care of the part outside this range. Solving the above ordinary differential equation with $y = r(x)$, we have

$$g(y) = \frac{y}{\delta_g} - \frac{y^2}{2\delta_g \theta U^*} + c_\alpha, \tag{16}$$

where c_α is determined if we impose a condition $g(\alpha) = \alpha$, in which case, we have

$$c_\alpha = \alpha\left[\frac{\delta_g - 1}{\delta_g} + \frac{\alpha}{2\delta_g \theta U^*}\right].$$

Since $r(x)$ satisfies

$$x = \frac{g^{-1}(x) + \delta_p r(x)}{1 + \delta_p},$$

we obtain

$$r(x) \;=\; -\frac{1}{\delta_p}\theta U^* + \frac{1}{\delta_p}\sqrt{(\theta U^*)^2 + 2\delta_g \theta U^*(c_\alpha - x)} + \frac{1 + \delta_p}{\delta_p}x,$$

or equivalently,

$$r(x) \;=\; -\frac{1}{\delta_p}\theta U^* + \frac{1}{\delta_p}\sqrt{(\theta U^* - \alpha)^2 + 2\delta_g \theta U^*(\alpha - x)} + \frac{1 + \delta_p}{\delta_p}x. \tag{17}$$

Since $g(y)$ is concave, the value function satisfies the second-order condition. Therefore, the optimality of g and r follows from the first order condition.

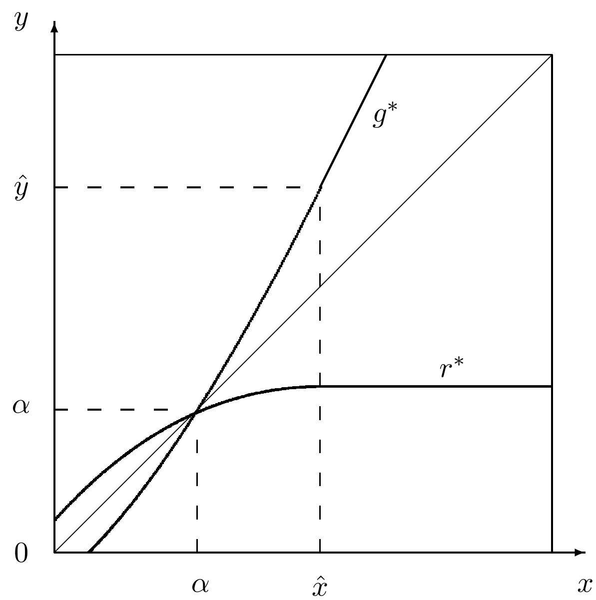

Figure 2: solution

Let us denote this pair of r and g solving the first order condition as $\hat{r}$ and $\hat{g}$.

Unfortunately, $(\hat{r}, \hat{g})$ has not given us an MPE (r^*, g^*) yet. Note that equation (15) is applicable only in the range of $[\inf r^*(x), \sup r^*(x)]$. Since r^* and g^* have to satisfy (17) and (16), respectively, in this range, $\inf r^*(x)$ and $\sup r^*(x)$ must be the same as $\inf \hat{r}(x)$ and $\sup \hat{r}(x)$. Therefore,

$$\inf r(x) = r(0) = -\frac{1}{\delta_p}\theta U^* + \frac{1}{\delta_p}\sqrt{(\theta U^* - \alpha)^2 + 2\delta_g \theta U^* \alpha},$$

and

$$\sup r(x) = r(\hat{x}) = \frac{1}{\delta_p}\left[-1 + \frac{\delta_g}{2(1+\delta_p)}\right]\theta U^* + \frac{1+\delta_p}{\delta_p}\alpha + \frac{1+\delta_p}{2\delta_p\delta_g\theta U^*}(\theta U^* - \alpha)^2,$$

where

$$\hat{x} = \alpha - \frac{\delta_g}{2(1+\delta_p)^2}\theta U^* + \frac{1}{2\delta_g \theta U^*}(\theta U^* - \alpha)^2$$

is calculated from $r'(x) = 0$. Note that for $x \in (\hat{x}, g(\theta U^*))$, we have $\hat{r}'(x) < 0$. Instead of having a negatively sloped portion, which violates Lemma 4.5, we replace it with a flat line, i.e.,

$$r^*(x) = \begin{cases} \hat{r}(x) & \text{if } x \leq \hat{x}, \\ r(\hat{x}) & \text{if } x > \hat{x}. \end{cases}$$

At the same time, the reaction function g of the private sector should also be modified.

$$g^*(y) = \begin{cases} \hat{g}(y) & \text{if } y \leq \hat{y}, \\ \frac{y + \delta_p r(\hat{x})}{1 + \delta_p} & \text{if } y > \hat{y}, \end{cases}$$

where

$$\hat{y} = \left(1 - \frac{\delta_g}{1 + \delta_p}\right) \theta U^*$$

is calculated from $\hat{g}(\hat{y}) = \hat{x}$. Note that g^* is still concave, and, therefore, the first-order condition is sufficient for optimality whenever this method is applicable. Note also that this transformation from $\hat{r}$ and $\hat{g}$ to r^* and g^* is applicable to the construction of an MPE only if $\hat{x} > \alpha$, or

$$\alpha < \left(1 - \frac{\delta_g}{1 + \delta_p}\right) \theta U^* = \varphi^{-1} \left(\frac{1 + \delta_p}{\delta_g}\right),$$

as desired.

Roughly speaking, the set of Markov perfect equilibria is bounded from above by the Ramsey outcome, but is bounded away from the KP outcome. In section 3, the set of subgame perfect equilibrium outcomes, which contains all Markov perfect equilibrium outcomes, converge to the Ramsey outcome. In contrast, the steady state of the Markov perfect equilibrium constructed above is bounded away from Ramsey and also from KP outcome. In fact, this outcome meets the lower bound of the set of Markov perfect equilibria characterized in section 4.

6 Concluding Remarks

In the dynamic policy game with no payoff-relevant link between periods, we have a result similar to the folk theorem. In contrast, in alternating move policy games, we sometimes have a sharp prediction. In particular, the suboptimal KP outcome is eliminated from the set of equilibria. Because of a payoff-relevant link between periods, the government can commit to a lower inflation rate and affect the private sector's response. Although the alternating move games capture the essence of asynchronous decision making processes, we need to investigate a more general form of asynchronous decision-making processes to see how robust our conclusion is.

Appendices

A Proof of Proposition 3.2

First of all, we have the following four inequalities for incentives of the private sector. $\Pi_{CL} \geq \Pi_{CL\setminus R}$ is equivalent to

$$3(1+\delta) \geq \delta p_L(4-\mu). \tag{18}$$

$\Pi_{DL} \geq \Pi_{DL\setminus R}$ is equivalent to

$$\mu + 3\delta - 1 \geq \delta p_L(4-\mu). \tag{19}$$

$\Pi_{CR} \geq \Pi_{CR\setminus L}$ is equivalent to

$$\delta p_R(4-\mu) \geq 3(1+\delta). \tag{20}$$

$\Pi_{DR} \geq \Pi_{DR\setminus L}$ is equivalent to

$$\delta p_R(4-\mu) \geq \mu + 3\delta - 1. \tag{21}$$

We divide the situation into three cases.
Case I. $V_{LC} > V_{LD}$, or equivalently, $V_{RC} > V_{RD}$.

It implies $p_L = p_R = 0$. Then (18) holds, while (20) does not, which implies $q_C = 0$. On the other hand, (19) holds if and only if $\mu \geq 1 - 3\delta$, and (21) holds if and only if $\mu \leq 1 - 3\delta$. Therefore, if $\mu > 1 - 3\delta$, then $q_D = 0$. But, in this case, using $V_{CL} = V_{DL} - 1$, we obtain $V_{LC} < V_{LD}$. A contradiction. So suppose $\mu < 1 - 3\delta$. Then $q_D = 1$ holds since (19) does not. It is verified that $q_C = 0$ and $q_D = 1$ imply $V_{LC} > V_{LD}$, and therefore, $(([C],[C]),([L],[R]))$ is the only MPE in this case if $\mu < 1-3\delta$, and no MPE exists for this case if $\mu > 1 - 3\delta$.
Case II. $V_{LC} < V_{LD}$, or equivalently, $V_{RC} < V_{RD}$.

It implies $p_L = p_R = 1$. Then (21) holds, while (19) does not, which implies $q_D = 1$. On the other hand, (18) holds if and only if $\mu \geq 1 - 3/\delta$, and (20) holds if and only if $\mu \leq 1 - 3/\delta$. Therefore, if $\mu > 1 - 3/\delta$, then $q_C = 0$. But this together with $q_D = 1$ implies $V_{LC} > V_{LD}$. A contradiction. Suppose $\mu < 1 - 3/\delta$. Then $q_C = 1$ holds. It is verified that $V_{LC} < V_{LD}$. Thus, in this case, $(([D],[D]),([R],[R]))$ is the only MPE if $\mu < 1-3/\delta$, and no MPE exists if $\mu > 1 - 3/\delta$.
Case III. $V_{LC} = V_{LD}$, or equivalently, $V_{RC} = V_{RD}$.

Case III-a. $\mu > 1 - 3/\delta$. In this subcase, $q_C = 0$ since (18) holds, and (20) does not for all (p_L, p_R). $V_{LC} = V_{LD}$ implies

$$\delta q_D(V_{DL} - V_{DR}) = 1 - \delta^2.$$

We have

$$V_{DL} - V_{DR} = 3(1 - \delta^2)$$

where we make use of $V_{LC} = V_{LD}$, $V_{RC} = V_{RD}$, and $V_{LC} - V_{RC} = 3(1 - \delta)$. Combining the above two equations, we obtain

$$q_D = \frac{1}{3\delta}.$$

In order for $(0, \frac{1}{3\delta})$ to be taken in an MPE, we need both (19) and (21). Therefore,

$$p_L \leq \frac{\mu + 3\delta - 1}{\delta(4 - \mu)},$$

and

$$p_R \geq \frac{\mu + 3\delta - 1}{\delta(4 - \mu)}.$$

Thus, this type of equilibrium exists if and only if

$$0 \leq \frac{\mu + 3\delta - 1}{\delta(4 - \mu)} \leq 1.$$

This inequality holds if and only if $\mu \geq 1 - 3\delta$.
Case III-b. $\mu < 1 - 3/\delta$. In this subcase, $q_D = 1$ since (21) holds, and (19) does not for all (p_L, p_R). $V_{LC} = V_{LD}$ implies

$$\delta(1 - q_C)(V_{CL} - V_{CR}) = 1 - \delta^2.$$

Using $V_{LC} - V_{RC} = 3(1 - \delta)$, we obtain $3\delta(1 - q_C) = 1$, or

$$q_C = 1 - \frac{1}{3\delta}.$$

We need

$$p_L \leq \frac{3(1 + \delta)}{\delta(4 - \mu)},$$

and

$$p_R \geq \frac{3(1 + \delta)}{\delta(4 - \mu)}.$$

This type of equilibrium exists if and only if $\mu \leq 1 - 3/\delta$ as desired.

This exhausts all the possibilities. Summarizing the result, we obtain the set of equilibria as we claimed.

B Proof of Lemma 4.5

Let $y_i = r(x_i)$ and $x_1 > x_2$. Since y_i is an optimal solution under state x_i,

$$\begin{aligned} & (1-\delta_g)\left[u(y_1,x_1)+\delta_g u(y_1,g(y_1))\right]+\delta_g^2\mathcal{U}(g(y_1)) \\ \geq \quad & (1-\delta_g)\left[u(y_2,x_1)+\delta_g u(y_2,g(y_2))\right]+\delta_g^2\mathcal{U}(g(y_2)) \end{aligned} \tag{22}$$

and

$$\begin{aligned} & (1-\delta_g)\left[u(y_2,x_2)+\delta_g u(y_2,g(y_2))\right]+\delta_g^2\mathcal{U}(g(y_2)) \\ \geq \quad & (1-\delta_g)\left[u(y_1,x_2)+\delta_g u(y_1,g(y_1))\right]+\delta_g^2\mathcal{U}(g(y_1)). \end{aligned} \tag{23}$$

By adding (22) to (23), and rearranging the terms, we have

$$u(y_1,x_1)+u(y_2,x_2) \geq u(y_2,x_1)+u(y_1,x_2).$$

Since $x_1 > x_2$,

$$\frac{u(y_1,x_1)-u(y_1,x_2)}{x_1-x_2} \geq \frac{u(y_2,x_1)-u(y_2,x_2)}{x_1-x_2}. \tag{24}$$

Since A2 implies that

$$\frac{\partial^2 u(y,x)}{\partial y \partial x} > 0,$$

(24) implies that $y_1 \geq y_2$ as desired. *Q.E.D.*

C Proof of Proposition 4.6

Let S be the set of all weakly increasing continuous-from-right functions over $[0,\overline{\alpha}]$. We can endow S with a weak topology: $f_n \to f$ if $f_n(x) \to f(x)$ pointwise whenever f is continuous at x. Note that S can be made a metric space, convex and compact with respect to the weak topology.

We shall construct a correspondence

$$\Phi : S \times S \to S \times S$$

where $\Phi = (\Phi_1, \Phi_2)$. Fix $(r,g) \in S \times S$. Given r, define

$$F(x) = (1+\delta_p)x - \delta_p r(x).$$

Recall that g is the inverse function of F, whenever F is invertible. Because $F^{-1}(y)$ in general has more than a single value, we need to do some work. Define

$$\underline{g}(y) = \inf\left\{g(y) : g \in S, \quad g(y) \in F^{-1}(x)\right\}$$

and

$$\overline{g}(y) = \sup\left\{g(y) : g \in S, \quad g(y) \in F^{-1}(x)\right\}.$$

Let

$$\Phi_2(r, g) = \mathrm{co}\left(\{\underline{g}, \overline{g}\}\right)$$

where co is the convex hull.

Given $g \in S$, the government solves the optimization problem. From Lemma 4.5, we know that any optimal response r must be weakly increasing. Since r jumps at x only if the government has multiple best response under x, we can always choose $r \in S$. Let

$$\Phi_1(r, g) = \mathrm{co}\left(\{r \in S : r \text{ is a best response to } g\}\right).$$

By construction, the correspondence Φ is upper hemi-continuous and convex valued. Since $S \times S$ is compact and convex, Glicksberg's fixed point theorem implies that there exists $(r, g) \in S \times S$ such that

$$(r, g) \in \Phi(r, g).$$

It is straightforward to prove that any fixed point of Φ is a Markov perfect equilibrium.

It remains to prove the existence of a steady state. Fix a Markov perfect equilibrium $(r, g) \in S \times S$. If r is a continuous function, the Brouwer's fixed point theorem implies that there exists $\alpha \in [0, \overline{\alpha}]$ such that

$$r(\alpha) = \alpha.$$

While r can be discontinuous at some $x \in [0, \overline{\alpha}]$, r must jump upward at x. To obtain the existence of a steady state, we can invoke the existence theorem by Roberts and Sonnenschein (1976) that extends the Brouwer's fixed point theorem to the class of mapping which has upward jump discontinuities. *Q.E.D.*

D Proof of Proposition 4.7

Suppose the contrary: $\exists \epsilon > 0$ and $\{\delta_{g,j}\}$ such that $\delta_{g,j} \to 1$, but

$$\inf_j \left[\sup\left\{\alpha' : g(\alpha') = \alpha'\right\} - \inf\left\{\alpha' : g(\alpha') = \alpha'\right\}\right] \geq \epsilon.$$

Let $\alpha_{sup} = \sup\{\alpha' : g(\alpha') = \alpha'\}$ and $\alpha_{inf} = \inf\{\alpha' : g(\alpha') = \alpha'\}$. By the Markov perfection, playing α_{sup} always is a Nash equilibrium. Similarly, playing α_{inf} always is a Nash equilibrium. Since $\alpha_{sup} > \alpha_{inf}$, $u(\alpha_{inf}, \alpha_{inf}) > u(\alpha_{sup}, \alpha_{sup})$. But, the long run payoff of switching from α_{sup} to α_{inf} and playing continuously α_{inf} is

$$(1 - \delta_{g,j})u(\alpha_{inf}, \alpha_{sup}) + \delta_{g,j}u(\alpha_{inf}, \alpha_{inf})$$

which is larger than $u(\alpha_{sup}, \alpha_{sup})$ for a sufficiently large $\delta_{g,j}$. This contradiction proves (9). *Q.E.D.*

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