A Theory of Money with Market Places*

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Abstract

This paper considers an infinitely repeated economy in which divisible fiat money is used to trade goods. The economy has many market places. In each period, each agent makes a production decision and chooses a market place. In each market place, agents are randomly matched to form a pair, and they trade their goods when both agree to do so. There exist various classes of stationary equilibria. In some equilibria, all the agents visit the same market place, while in others, market places are specialized, i.e., at most one type of good is traded in each market place. In some equilibria, each good is traded at a single price, while in others, every good is traded at two different prices. Each class itself consists of equilibria with infinitely many price and welfare levels. However, it is shown that only the efficient single price equilibria with specialized market places are evolutionarily stable. An inefficient equilibrium is upset by the mutants who visit inactive market places to establish a more efficient trading pattern than before. An extension to the economy with multiple currencies is also examined.

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1 Introduction

In a transaction, one needs to have what his trade partner wants. However, it is hard for, say, an economist who wants to have her hair cut to find a hairdresser who wishes to learn economics. In order to mitigate this problem of a lack of double coincidence of wants, money is often used as a medium of exchange. If there is a generally acceptable good called money, then the economist can divide the trading process into two: first, she teaches economics students to obtain money, and then finds a hairdresser to exchange money for haircut. The hairdresser accepts the money since he can use it to obtain what he wants. Money is accepted by many people as it is believed to be accepted by many.

Focusing on this function of money, Kiyotaki and Wright [10] formalize the process of monetary exchange. In their model, agents are randomly matched to form a pair and trade their goods when they both agree to do so. This and the subsequent models, called the search theoretic models of money, have laid a foundation of monetary economics. The purpose of the present paper is to further develop the foundation by introducing the concept of *market places* into the model with divisible fiat money presented by Green and Zhou [6].

Market places are the places which agents choose to visit to meet trade partners. They capture the following two aspects of actual trading processes. First, matching rarely occurs in a completely random fashion in the real economy. People go to a fish market as buyers to buy fish; many potential workers use particular channels for job openings rather than simply walking on streets to meet potential employers. There are market places where agents look for their trade partners.

The second aspect is related to price competition. In order for price competition to take place, there must be a possibility that a price cut leads to an increase in sales. The standard search model incorporates this possibility only in a limited way. While a seller may increase sales per visit by lowering its price, it cannot increase the number of visits itself. In other words, even if the seller cuts the price, it cannot differentiate itself from its competitors and attract more customers. If there are many market places, sellers can differentiate themselves from their competitors by visiting an inactive market place. It turns out that this function of market places activates the price adjustment mechanism in the present analysis.

In order to see how market places function, let us now briefly explain the model and the results of the paper. Roughly speaking, the present model is described as follows. It has an infinite number of periods and a continuum of infinitely lived agents. There are infinitely many market places, each of which consists of two physically identical sides, A and B. In each period, all the agents simultaneously decide whether to produce a unit of goods or not and choose a market place and one of its sides. In each market place, the agents on side A are matched with those on side B in a random fashion to form pairs, with the long side being rationed.[1] When two agents are matched and find that one of them holds the good that the other wishes to consume, they negotiate the price, which is modeled as a simultaneous offer game.

We adopt two approaches for the main analysis, an equilibrium approach and an evolutionary approach. We first examine stationary equilibria of this model. There exist various classes of stationary equilibria. Equilibria can be classified based upon two characteristics, the degree of specialization of market places, i.e., the number of goods traded in one market place, and the degree of price dispersion, especially, whether each good is traded at a single price or not. In some equilibria, all the agents visit the same market place, while in others, market places are completely specialized, i.e., at most one type of good is traded in each market place. In some equilibria, each good is traded at a single price, while in others, each good is traded at two different prices. Each class itself consists of a continuum of equilibria which correspond to different price and welfare levels.

As we mentioned above, the equilibrium approach admits a multitude of outcomes. The reason is the following. Suppose, for the sake of argument, that all the sellers of the same type of good go to the same market places. In the present model, the only way that a seller can increase the matching probability is to switch to an empty market place to differentiate itself from other sellers, but no buyer visits such a place if no seller is expected to visit there. Therefore, no unilateral deviation to an empty market place is profitable. In other words, the equilibrium approach cannot have agents utilize empty places to start a new transaction pattern, including the one with a price cut.

The evolutionary approach overcomes this coordination problem. It allows a small group of agents, or mutants, to jointly visit inactive (empty of

[1] These sides are introduced to simplify some of the subsequent analysis. One story would be that there are many beaches, and some agents row boats to meet those waiting for them along the seashore. Of course, one should not take the present formulation too literally, just as we should not take the standard random matching model too literally.

thin) market places and start a new transaction pattern. An equilibrium is said to be evolutionarily stable if no group of mutants fares better than the original population. It is shown that only efficient single price equilibria with complete specialization are evolutionarily stable. An inefficient equilibrium is upset by the mutants who visit inactive market places to establish a more efficient trading pattern than before.

The present paper serves a microfoundation for the trading post approach. This approach is initiated by Shapley and Shubik [18], and applied to a situation with fiat money by Hayashi and Matsui [7]. Trading posts are the places in each of which a prespecified pair of goods are traded. People submit their goods to the designated trading posts. The goods they submit to one side of a post are traded with the other type of goods submitted to the opposite side. Agents obtain the goods on the opposite side in proportion to the amount they submitted. The trading mechanism at trading posts is put in a black box. On the other hand, the present paper explicitly models the trading processes. At the same time, it prepares sufficiently many market places that can be used for transactions, but does not specify which place is used for which goods to be traded. Specialization of market places may emerge endogenously. It is verified that the evolutionarily stable outcome of the present model corresponds to the stationary equilibrium examined in Hayashi and Matsui.[2]

Some mention has to be made of the existing search theoretic models of money. In the beginning, these models (e.g., Kiyotaki and Wright [10]) assume indivisible commodities and fiat money, if any, mainly due to the analytical difficulty of tracking inventory as different agents have different experiences. Avoiding this difficulty, Trejos and Wright [20] and Shi [19] addressed the issues related to price levels. In order to do so, Trejos and Wright introduced divisible commodities, while Shi presented a model in which each household can simultaneously engage in infinitely many transaction activities. However, each transaction involves an indivisible unit of money. Green and Zhou [6] presented a model with divisible fiat money. They partially succeeded in solving it, restricting their analysis to a certain class of equilibria.

[2] Iwai's trading zone model [9] is also related to the present paper. Given the number of commodities n, each agent chooses one of $n(n-1)/2$ trading zones in which random matching takes place. The matching probability in a certain zone is proportional to the number of agents visiting the zone. Each agent can hold one unit of indivisible commodities storable with some costs. He examined which commodity becomes a medium of exchange.

A new problem arised in Green and Zhou: there exist a continuum of equilibria with different price and welfare levels. Green and Zhou expanded the frontier of the search theoretic models of money, but at the same time, it revealed the fundamental problem of indeterminacy associated with these models.

A crucial reason for this indeterminacy is that, as we mentioned above, the probability of matching is exogenous, and therefore, say, a seller cannot attract more customers by lowering its price even if there is excess supply in the market. The existence of market places allows the possibility of changes in the probability of matching so that the price adjustment mechanism works.

Models with endogenous matching are not new in other fields. Directed search models in labor economics and local interaction models in evolutionary game theory both have dealt with endogenous matching. Among them, the closest to the present paper in terms of formulation of matching technology are Moen [14] in labor economics, and Mailath, Samuelson, and Shaked [11] in evolutionary game theory. Moen constructed a model in which firms with different wage offers are assigned to different submarkets, and workers choose a submarket to be matched with a firm in the same submarket. The assignment of firms to submarkets is carried out by an exogenous mechanism. Workers then observe the list of prices attached to these submarkets before they choose a submarket to visit. There is a gap between Moen's approach and ours. In the sense that workers know where to go in order to obtain a job at a specific wage, Moen's approach is closer to Peters [16], who constructed a model in which sellers publicly declare prices before buyers choose which seller to visit, than search theoretic models in which one can infer but cannot observe the price before matching. On the other hand, our approach follows the tradition of search theory in the sense that the price is observed only after traders are matched with each other.

Mailath *et al.* considered a situation in which players decide to go to certain locations, in which they are randomly matched to play a prespecified game. Search theoretic models of money have a richer structure than simple finite games analyzed therein as well as addtional complexity due to changes in agents' money holdings.[3]

[3] Recently, Corbae, Temzelides, and Wright [2] wrote a paper on monetary economics with endogenous matching. They considered a situation in which agents are matched to form pairs in the way formalized by Gale and Shapley [5], i.e., the concept of core is used to find optimal matching. In this sense, matching is not only endogenous but also non-random. They mixed cooperative and noncooperative concepts in their analysis in

The rest of the paper is organized as follows. Section 2 presents our framework. Section 3 defines and characterizes stationary equilibria, which is followed by the welfare analysis and comments on the effects of short-run monetary policies. Section 4 identifies the essentially unique evolutionarily stable equilibrium. Section 5 extends the model to the economy with multiple currencies and discusses some issues associated with it. Section 6 concludes the paper. Appendices for lengthy proofs are also attached.

2 Model

We consider an infinite repetition of an economy which is inhabited by a continuum of agents with measure one. Time is discrete and indexed by $t = 1, 2, \ldots$. There are K types of agents, $1, \ldots, K$. The generic element is denoted by k. Assume $K \geq 3$. The mass of each type is $1/K$. There are K types of commodity goods, $1, \ldots, K$, and good 0, or fiat money. An agent of type k obtains utility $u > 0$ if he consumes one unit of good k. Every commodity good is perishable and indivisible. He can produce at most one unit of good $k + 1 \pmod K$ in each period. Its production cost is zero. We assume that agents do not produce goods unless they expect to sell the goods with a positive probability.[4] On the other hand, fiat money is non-perishable and divisible. Each agent can hold any amount of fiat money with no cost. We assume that agents immediately discard fiat money which they never expect to use.[5] M is the total nominal stock of fiat money.

There are countably many *market places*, indexed by $z = 1, 2, 3, \ldots$. Each market place has two physically identical sides, A and B.

Each period consists of the following four stages.

Stage 1: Agents simultaneously decide whether to produce goods or not and choose a market place and one of its sides.

Stage 2: In each market place, a random matching takes place. The matching technology is frictionless, though the long side is rationed. Also, the

the sense that core is used within each period, while across periods, a noncooperative equilibrium concept is adopted.

[4]This assumption corresponds to the existence of an infinitesimal production cost. It eliminates equilibria in which gift giving occurs between anonymous agents.

[5]This assumption corresponds to the existence of an infinitesimal holding cost of fiat money.

matching is uniform. Formally speaking, if the measure of the agents visiting side A is n_A, that of the agents visiting side B is n_B, and among those visiting side B are the agents who belong to set S with its measure being n_S, then the probability that an agent visiting side A meets someone in S is $\min\{n_S/n_A, n_S/n_B\}$.[6]

Stage 3: If a type k agent and a type $k+1 \pmod K$ agent are matched, the type k agent offers a price p_S, and the type $k+1 \pmod K$ agent bids a price p_B. The type of each agent is observable to his partner, but not his money holdings. We assume that $p_S = \infty$ if the type k agent did not produce his good at Stage 1. No trade takes place in any other type of matching, and in such a case, agents do not make any further move, i.e., they skip Stage 4.

Stage 4: If $p_S \leq p_B$, then the type k agent exchanges his good for p_S units of fiat money, and the type $k+1 \pmod K$ agent exchanges p_S units of fiat money for the good, and consumes it.[7]

From now on, we say "a seller meets a buyer" when a type k agent meets a type $k+1 \pmod K$ agent.

The subsequent analysis uses Markov strategies, according to which actions depend only on the current money holdings of the agent in question.[8] Formally, a *Markov strategy* is defined to be a triple $\sigma = (\lambda, o, \beta)$ where

- $\lambda : \mathbb{R}_+ \to \mathbb{N} \times \{A, B\}$: a *location strategy*;
- $o : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$: an *offer strategy*; and
- $\beta : \mathbb{R}_+ \to \mathbb{R}_+$: a *bidding strategy*.

In this expression, $\lambda(\eta) = (z, s)$ implies that the agent who takes λ and holds η units of money chooses side s of market place z. Production decisions are

[6] Although we can construct a one-to-one and onto mapping between two sets of agents with different measures, we assume that rationing still occurs if the measure of one set is different from that of the other.

[7] The subsequent analysis will not be affected at all even if we change the rule on which price to use as long as the price is between p_S and p_B.

[8] We use the word "Markov" more restrictively than used in some other contexts in the sense that Markov strategy in our definition is independent of the current location and the current distribution of other agents' money holdings. However, even if such an alternative definition is adopted, the subsequent results remain unchanged.

reflected in offer strategies, i.e., $o(\eta) = \infty$ implies that the agent does not produce, while $o(\eta) = p \leq \infty$ implies that the agent produces a good and offers p if he meets a buyer. We assume $\beta(\eta) \leq \eta$, i.e., the buyer cannot bid beyond his current money holdings. The set of all Markov strategies is denoted by Σ. In the sequel, we allow deviating agents to take full-fledged strategies. A full-fledged strategy is a function from the set of the entire personal histories, into the set of appropriate actions.[9] The set of all strategies is denoted by $\bar{\Sigma}$.

Moreover, we impose symmetry across types on Markov strategies in the subsequent arguments.[10] However, we allow different agents of the same type to take different strategies. Henceforth, we represent a symmetric strategy profile by the strategy of type k agents. For example, for a location strategy λ, $\lambda(\eta) = (1, A)$ means that the agents of all types with money holdings η visit $(1, A)$, and $\lambda(\eta) = (K + k + 1, B)$ means that type k agents with η visit side B of market place $K + (k + 1 \pmod K)$, type $k + 1$ agents with η visit side B of market place $K + (k + 2 \pmod K)$, and so on.

We denote by μ a distribution on money holdings and strategies: $\mu(\{\eta\}; \{\sigma\})$ is the fraction of the agents who take σ and hold η units of money, which we write $\mu(\eta; \sigma)$ whenever it causes no confusion. Notice that we have extended the notion of symmetry, imposing it on distributions. Given μ, μ_Σ is its marginal distribution on strategies, i.e., $\mu_\Sigma(\Sigma') \stackrel{\text{def}}{=} \mu(\mathbb{R}_+; \Sigma')$ is the fraction of the agents taking strategies in $\Sigma' \subset \Sigma$. Similarly, μ_H is its marginal distribution on money holdings.

The transition of an agent's money holdings η is straightforward. Suppose that the agent takes σ. If he meets a seller with $(\sigma', \eta') = ((\lambda', o', \beta'), \eta')$, and if $\beta(\eta) \geq o'(\eta')$, then his money holdings become $\eta - o'(\eta')$. If, on the other hand, the agent meets a buyer with $(\sigma', \eta') = ((\lambda', o', \beta'), \eta')$, and if $\beta'(\eta') \geq o(\eta)$, then his money holdings become $\eta + o(\eta)$. Otherwise, η remains unchanged.

[9] A personal history contains his past transaction records, especially, the current money holdings and the type of the current trade partner, and some observable aggregate data. We do not specify which aggregate data are observable since it does not affect the subsequent analysis.

[10] We call a Markov strategy profile $\sigma = \left(\sigma^1, \ldots, \sigma^K\right)$ *symmetric* if, when all type k agents take σ^k and money holdings distributions of all types are identical, the probability that a type k agent is matched with a type $k + i \pmod K$ agent is the same as the probability that a type $k + 1 \pmod K$ agent is matched with a type $k + i + 1 \pmod K$ agent for any $i = 1, \ldots, K - 1$ and offer prices and bid prices are common.

Given t, each agent tries to maximize the discounted average of future stage payoffs, i.e.,

$$E\left[(1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}u_{\tau}\;\middle|\;\Omega_t\right],$$

where $\delta \in (0,1)$ is a common discount factor, u_τ is u if he obtains his consumption good at period τ, and zero otherwise, and Ω_t is the information available at period t. In particular, we denote the above expression by $V^t(\sigma,\eta^t;\mu^t)$ if he takes σ with η^t, money holdings at period t, and the distribution at period t is μ^t. Hereafter, we suppress superscripts t's to write $V(\sigma,\eta;\mu)$ whenever it causes no confusion.

3 Stationary Equilibria

3.1 Equilibrium Concept

To begin with, we define stationary distribution. A distribution μ is said to be *stationary* if μ is transformed into μ when all the agents do not revise their strategies, and their money holdings follow the above transition rule.[11] We are now in the position to define our equilibrium concept.

Definition 1 A stationary distribution μ is a *stationary (symmetric Markov perfect) equilibrium* iff only Markov strategies are taken, and no agent has an incentive to deviate with any money holdings, i.e., for all σ in the support of μ_Σ, all $\tilde{\sigma} \in \bar{\Sigma}$, and all $\eta \in \mathbb{R}_+$,

$$V(\sigma,\eta;\mu) \geq V(\tilde{\sigma},\eta;\mu).$$

This definition implies that an equilibrium should satisfy a requirement similar to subgame perfection with regard to money holdings. Note that in any Markov perfect equilibrium, the agent produces his production good if and only if he expects to meet a buyer whose bid is no more than his intended offer with a positive probability.

[11] This definition implies that agents never discard money on the path of a stationary distribution.

3.2 Equilibria with No Specialization

First of all, there exist equilibria in which only one market place is used. These equilibria correspond to those found in the complete random matching model of Green and Zhou [6]. Suppose that all the agents go to the same market place, and moreover, they are evenly distributed between sides A and B, and offer and bid a common price p, if possible. Then no agent has an incentive to change his location strategy since a visit to another market place gives him no utility, and the situation is exactly the same if he visits the other side. It is also verified that they have no incentive to change their offer and bidding strategies if the price p is sufficiently high.

In this class of equilibria, each agent may end up in selling a good even if he has a sufficient amount of money to buy his consumption good. Consequently, the support of the distribution of money holdings is a countable set.[12] There are a continuum of equilibria with different price and welfare levels.

3.3 Single Price Equilibria with Complete Specialization

In this subsection and the next, we consider equilibria with completely specialized market places, i.e., those in which at most one type of good is traded in each market place.

Given a price level $p > 0$, a *single price equilibrium with complete specialization and with* p (henceforth, a *p-SPE*) is a stationary equilibrium in which every good is traded at p. The canonical p-SPE, μ_p, is defined as follows:

- $\mu_p(0; \sigma_p) = 1 - m$,
- $\mu_p(p; \sigma_p) = m$,

where $m = M/p$ is the total real stock of fiat money, and $\sigma_p = (\lambda_p, o_p, \beta_p)$ is a Markov strategy such that

- $\lambda_p(\eta) = \begin{cases} (k, B) & \text{if } \eta \geq p, \\ (k+1, A) & \text{if } \eta < p, \end{cases}$

[12] The reader should refer to Green and Zhou for the formal description of this class of equilibria.

- $o_p(\eta) = \begin{cases} p & \text{if } \eta < p, \\ \infty & \text{if } \eta \geq p, \end{cases}$

- $\beta_p(\eta) = \begin{cases} p & \text{if } \eta \geq p, \\ \eta & \text{if } \eta < p. \end{cases}$

In short, non-money holders go to side B of an appropriate market place to meet the buyers of their production goods. While money holders, who have p units of money, go to side A of an appropriate market place to meet the sellers of their consumption goods. Figure 1 illustrates who goes where on the equilibrium path of the canonical p-SPE, and Figure 2 gives the transition of each agent's money holdings. Note that μ_p is stationary. We now state and prove that these canonical distributions are indeed stationary equilibria.

Theorem 1 For all $p > M$, and all δ, the canonical p-SPE μ_p is a stationary equilibrium.

Proof: We denote $V(\sigma_p, \ell p; \mu_p)$ by V_ℓ for $\ell \in \mathbb{N}_+$.[13] We divide the proof into two cases:

Case 1: $m \geq \frac{1}{2}$.

In this situation, the sellers are lying on the short side. Therefore, the buyers are rationed (except in the case of $m = 1/2$), while the sellers are not. Let $r = \frac{1-m}{m}$. Then we have the following value functions:

$$V_0 = \delta V_1,$$
$$V_\ell = r\left((1-\delta)u + \delta V_{\ell-1}\right) + (1-r)\,\delta V_\ell, \quad \ell \geq 1.$$

Solving this system of equations, we obtain

$$V_\ell = r\left(1 - \left[\frac{\delta r}{1-\delta+\delta r}\right]^{\ell-1} \frac{\delta r}{1+\delta r}\right) u, \quad \ell \geq 0.$$

The only incentive compatibility condition that we need to verify is the one under which no money holder becomes a seller. It is given by

$$V_\ell \geq \delta V_{\ell+1}, \quad \ell \geq 1,$$

[13] For general η, we have $V(\sigma_p, \eta; \mu_p) = V\left(\sigma_p, \lceil \frac{\eta}{p} \rceil p; \mu_p\right)$ where $\lceil x \rceil$ is the integer part of x.

which is equivalent to

$$(1-\delta)\, r \left(1 - \left[\frac{\delta r}{1-\delta+\delta r}\right]^{\ell}\right) u \geq 0, \quad \ell \geq 1.$$

Since we have $\delta < 1$ and $r > 0$, this inequality holds.

Case 2: $m < \frac{1}{2}$.

In this situation, the buyers are lying on the short side. Therefore, the sellers are rationed, while the buyers are not. Let $r = \frac{m}{1-m}$. Then, we have the following value functions:

$$\begin{aligned} V_0 &= r\delta V_1 + (1-r)\,\delta V_0, \\ V_\ell &= (1-\delta)\, u + \delta V_{\ell-1}, \quad \ell \geq 1. \end{aligned}$$

Solving this system of equations, we obtain

$$V_\ell = \left(1 - \delta^{\ell} \frac{1}{1+\delta r}\right) u, \quad \ell \geq 0.$$

The only incentive compatibility condition that we need to verify is the one under which no money holder becomes a seller. It is given by

$$V_\ell \geq r\delta V_{\ell+1} + (1-r)\,\delta V_\ell, \quad \ell \geq 1,$$

which is equivalent to

$$(1-\delta)\left(1-\delta^{\ell}\right) u \geq 0, \quad \ell \geq 1.$$

Thus, this condition holds. ∎

3.4 Dual Price Equilibria with Complete Specialization

In the present model, since there can be more than one market places for a specific transaction, there is no *a priori* reason that a single price prevails. In fact, there are equilibria in which the same goods are traded at different prices. The simplest class of such equilibria is given below.

A *dual price equilibrium* is a stationary equilibrium in which each good is traded at two different prices. In particular, given $p > 0$ and an integer

$n \geq 2$, we consider a *dual price equilibrium with complete specialization and with two price levels p and np* (henceforth, we call it *(p, np)-DPE*) which has two classes of viable market places, *low-price* markets, in which goods are traded at price p, and *high-price* markets, in which goods are traded at price np.

In such an equilibrium, the low-price markets are in excess demand; for if not, all the buyers go there. On the other hand, the high-price markets are in excess supply; for if not, all the sellers go there.[14]

The canonical (p, np)-DPE, $\mu_{(p,np)}$, is given in Table 1 together with the following description. In Table 1, the entry for (σ, η) $(\sigma = \sigma_1, \sigma_n;\ \eta = 0, p, np)$ is $\mu_{(p,np)}(\sigma, \eta)$, e.g., $\mu_{(p,np)}(\sigma_n, 0) = h_{0n}$. These values are determined in the sequel.

$\sigma \setminus \eta$	0	p	np
σ_1	h_{01}	h_1	0
σ_n	h_{0n}	0	h_n

Table 1: Canonical (p, np)-DPE

In this distribution, $\sigma_i = (\lambda_i, o_i, \beta)$ $(i = 1, n)$ is given by

- $\lambda_1(\eta) = \begin{cases} (K + k, B) & \text{if } \eta \geq np, \\ (k + 1, A) & \text{if } np > \eta > \ell^* p, \\ (k, B) & \text{if } \ell^* p \geq \eta \geq p, \\ (k + 1, A) & \text{if } \eta < p, \end{cases}$

- $\lambda_n(\eta) = \begin{cases} (K + k, B) & \text{if } \eta \geq np, \\ (k + 1, A) & \text{if } np > \eta > \ell^* p, \\ (k, B) & \text{if } \ell^* p \geq \eta \geq p, \\ (K + k + 1, A) & \text{if } \eta < p, \end{cases}$

- $o_1(\eta) = \begin{cases} p & \text{if } \eta < p, \text{ or } \ell^* p < \eta < np, \\ \infty & \text{otherwise,} \end{cases}$

[14] In labor economics, Montgomery [15] shows the existence of equilbria with wage dispersion by using a similar idea.

- $o_n(\eta) = \begin{cases} np & \text{if } \eta < p, \\ p & \text{if } \ell^* p < \eta < np, \\ \infty & \text{otherwise,} \end{cases}$

- $\beta(\eta) = \begin{cases} np & \text{if } \eta \geq np, \\ p & \text{if } np > \eta \geq p, \\ \eta & \text{if } \eta < p, \end{cases}$

for some integer ℓ^* such that $n > \ell^* \geq 1$. Given p, n, and δ, the value of ℓ^* is determined in Appendix A so as to satisfy agents' incentive constraints off the equilibrium path. In this description, market places $1, \ldots, K$ correspond to the low-price markets, while $K+1, \ldots, 2K$ correspond to the high-price markets.

The above strategies look complicated partly because they specify agents' behavior not only on the equilibrium path, but also off the equilibrium path. On the equilibrium path, we have

$$\sigma_1(0) = ((k+1, A), p, \cdot),$$
$$\sigma_1(p) = ((k, B), \cdot, p),$$

and

$$\sigma_n(0) = ((K+k+1, A), np, \cdot),$$
$$\sigma_n(np) = ((K+k, B), \cdot, np).$$

There are two groups of agents in the market. The first group consists of the agents who take σ_1, using the low-price markets. While the second group consists of those who take σ_n, using the high-price markets. Each agent in both groups probabilistically alternates between a buyer and a seller. Figure 3 shows where agents go in this equilibrium, and Figure 4 gives the transition of each agent's money holdings. Some agents are rationed and stay in the same state, which is omitted in the figure.

We now examine some necessary conditions for the canonical (p, np)-DPE to form a stationary equilibrium under any $\delta \in (0,1)$. First of all, since both $(\sigma_1, 0)$ and $(\sigma_n, 0)$ are in the support of $\mu_{(p,np)}$, non-money holders must be indifferent between the low-price markets and the high-price markets. Let $\tilde{r} = h_{01}/h_1$ be the ratio of the sellers to the buyers in the low-price markets, and $\hat{r} = h_n/h_{0n}$ be the ratio of the buyers to the sellers in the high-price markets. As is mentioned above, both $\tilde{r}$ and $\hat{r}$ are strictly between zero and one. Let $\tilde{V}_0$ be the value for the non-money holders who take σ_1. After some calculations, it is written as

$$\tilde{V}_0 = \frac{\delta \tilde{r}}{1+\delta \tilde{r}} u.$$

Similarily, let $\hat{V}_0$ be the value for the non-money holders who take σ_n. Then it is written as

$$\hat{V}_0 = \frac{\delta \hat{r}}{1+\delta \hat{r}} u.$$

The condition $\tilde{V}_0 = \hat{V}_0$ implies $\tilde{r} = \hat{r}$. Henceforth, we denote $\tilde{V}_0$ and $\hat{V}_0$ by V_0 and $\tilde{r}$ and $\hat{r}$ by r.

Next, the agents with p units of money must go to the low-price markets as buyers according to the equilibrium strategy. Consider the following deviation. An agent goes to the low-price market as a seller to save money up to np, goes to the high-price markets as a buyer, and returns to the equilibrium strategy. No agent has an incentive for such a deviation if and only if

$$\frac{r}{1-\delta(1-r)} \left((1-\delta) u + \delta V_0 \right) \geq \delta^{n-1} \left((1-\delta) u + \delta V_0 \right),$$

which is equivalent to

$$r \geq \frac{\delta^{n-1}}{1+\delta+\cdots+\delta^{n-1}}.$$

This inequality holds for any δ close to 1 if and only if $r \geq 1/n$ holds.

On the other hand, agents with np units of money must go to the high-price markets as buyers according to the equilibrium strategy. Consider the following deviation. An agent with np units of money goes to the low price market as a buyer until he spends all the money and then returns to the equilibrium strategy. No agent has such an incentive if and only if

$$(1-\delta) u + \delta V_0 \geq r \left(1 - \left[\frac{\delta r}{1-\delta(1-r)} \right]^n \right) u + \left[\frac{\delta r}{1-\delta(1-r)} \right]^n V_0.$$

After tedious calculations, it is verified that the above inequality holds for any δ close to 1 if and only if we have $r \leq 1/n$.

To sum up, a necessary condition that the canonical DPE is a stationary equilibrium for all δ is $r = 1/n$. Moreover, this necessary condition is proven to be sufficient. In other words, it is verified that, for all p, all n, and all δ, there exists ℓ^* such that σ_1 and σ_n are the best responses to $\mu_{(p,np)}$ if and only if $r = 1/n$ holds. See Appendix A for the detail of the proof.

When $r = 1/n$, we have

$$\frac{M}{p} = h_1 + nh_n = \frac{n}{n+1},$$

so the low price is determined uniquely as $\frac{1}{n}(n+1)M$.

Theorem 2 For all integer $n \geq 2$, and all δ, the canonical $\left(\frac{1}{n}(n+1)M, (n+1)M\right)$-DPE $\mu_{\left(\frac{1}{n}(n+1)M,(n+1)M\right)}$ is a stationary equilibrium.[15]

3.5 Welfare Analysis

This subsection examines the welfare of various stationary equilibria. We define

$$V(\mu, \hat{\mu}) \stackrel{\text{def}}{=} \int_{\bar{\Sigma}\times\mathbb{R}_+} V(\sigma, \eta; \hat{\mu})\, d\mu(\sigma, \eta),$$

and

$$W(\mu) \stackrel{\text{def}}{=} V(\mu, \mu).$$

We regard W as the welfare of the economy. In other words, welfare is assumed to be measured by the average value of all the agents. Moreover, W is used as the criterion of efficiency. Formally, we call a stationary distribution μ *efficient* if μ maximizes $W(\mu)$. The maximum value of $W(\mu)$ is $\frac{1}{2}u$ due to the assumptions on production and matching technologies, according to which one cannot produce and consume in the same period, and therefore, in each period, at most a half of the entire population obtain u.

First, if only one market place is used as in the case of no specialization, then the probability of a type k agent being matched with an agent of either type $k-1$ or $k+1$ is at most $2/K$. Furthermore, at most only a half of the

[15] Strictly speaking, the canonical (p, np)-DPE is not *a distribution*, but *a class of distributions* since it does not completely specify ℓ^* and $\mathbf{h} = (h_{01}, h_{0n}, h_1, h_0)$. Therefore, the precise statement is the following: for all integer $n \geq 2$, and all δ, there exists ℓ^* such that the canonical $\left(\frac{1}{n}(n+1)M, (n+1)M\right)$-DPE $\mu_{\left(\frac{1}{n}(n+1)M,(n+1)M\right)}$ with ℓ^* and $\mathbf{h} = (\frac{1}{n+1}h, \frac{n}{n+1}(1-h), \frac{n}{n+1}h, \frac{1}{n+1}(1-h))$ $(h \in (0,1))$ is a stationary equilibrium.

matched agents can obtain u. Therefore, the welfare for this case is at most $\frac{1}{K}u$.

Next, we calculate the welfare of canonical single and dual price equilibria with complete specialization.

(i) p-SPE:

(a) Case 1: $m \geq \frac{1}{2}$, i.e. $p \leq 2M$.

$$\begin{aligned} W(\mu_p) &= (1-m)V_0 + mV_1 \\ &= (1-m)u. \end{aligned}$$

(b) Case 2: $m \leq \frac{1}{2}$, i.e. $p \geq 2M$.

$$W(\mu_p) = mu.$$

In particular, $W(\mu_p)$ attains the maximum value when $p = 2M$, i.e.,

$$W(\mu_{2M}) = \frac{1}{2}u.$$

(ii) $\left(\frac{1}{n}(n+1)M, (n+1)M\right)$-DPE:

$$\begin{aligned} W\left(\mu_{\left(\frac{1}{n}(n+1)M,(n+1)M\right)}\right) &= (h_{01} + h_{0n})V_0 + h_1V_1 + h_nV_n \\ &= \frac{1}{n+1}u. \end{aligned}$$

To sum up, the canonical $2M$-SPE is efficient, while the other canonical single price equilibria and all the canonical dual price equilibria are inefficient.

Theorem 3 The canonical $2M$-SPE is efficient.

Is the reverse of the above statement true? That is, can we say that only single price equilibria with complete specialization, like $2M$-SPE, is efficient? Without any qualification, the answer is no. Indeed, efficient equilibria are not always the ones with complete specialization. The following equilibrium with partially specialized market places serves a counter example.

Let μ be defined as follows:

- $\mu(0;\sigma) = \frac{1}{2}r$,
- $\mu(p;\sigma) = \frac{1}{2}$,
- $\mu(3p;\sigma) = \frac{1}{2}(1-r)$,

where $\sigma = (\lambda, o, \beta)$ is a Markov strategy such that

- $\lambda(\eta) = \begin{cases} (k-1, A) & \text{if } \eta \geq 2p, \\ (k, B) & \text{if } 2p > \eta \geq p, \\ (k+1, A) & \text{if } \eta < p. \end{cases}$

- $o(\eta) = \begin{cases} 2p & \text{if } 2p > \eta \geq p, \\ p & \text{if } \eta < p, \\ \infty & \text{otherwise,} \end{cases}$

- $\beta(\eta) = \begin{cases} 2p & \text{if } \eta \geq 2p, \\ p & \text{if } 2p > \eta \geq p, \\ \eta & \text{if } \eta < p. \end{cases}$

Figure 5 shows where agents go in this equilibrium.

It is verified that, given δ, μ is a stationary equilibrium if r is sufficiently small. At the same time, μ is efficient since a half of the agents consume at every period.

However, this equilibrium is not robust in the sense that it disappears as an equilibrium if agents are sufficiently patient. To see it, consider a Markov strategy $\tilde{\sigma} = (\tilde{\lambda}, \tilde{o}, \tilde{\beta})$ defined as follows:

- $\tilde{\lambda}(\eta) = \begin{cases} (k-1, A) & \text{if } \eta \geq 4p, \\ (k, B) & \text{if } 4p > \eta \geq p, \\ (k+1, A) & \text{if } \eta < p, \end{cases}$

- $\tilde{o}(\eta) = \begin{cases} 2p & \text{if } 4p > \eta \geq p, \\ p & \text{if } \eta < p, \\ \infty & \text{otherwise,} \end{cases}$

- $\tilde{\beta}(\eta) = \begin{cases} 2p & \text{if } \eta \geq 4p, \\ p & \text{if } 4p > \eta \geq p, \\ \eta & \text{if } \eta < p. \end{cases}$

Note that replacing σ with $\tilde{\sigma}$, an agent can increase the frequency of consumption, though he has to defer the opportunities of consumption. Therefore, given r, if δ is sufficiently large, it is more profitable to take $\tilde{\sigma}$ than σ.

In order to present a formal result, we first define the following.

Definition 2 A strategy distribution μ is a *robust equilibrium* (with respect to discounting) if there exists $\bar{\delta}$ such that μ is a stationary equilibrium for any $\delta \in (\bar{\delta}, 1)$.

We then state the following result of which proof is relegated to Appendix B.

Theorem 4 Any efficient robust equilibrium is a $2M$-SPE.

Note that there are other robust equilibria that are inefficient. First, all the canonical p-SPE's with $p > M$ are robust. It is also verified that for any integer $n \geq 2$, there exists a robust canonical $(\frac{1}{n}(n+1)M, (n+1)M)$-DPE.[16]

3.6 Monetary Policies: Short-run

The effectiveness of monetary policies is one of the central issues in monetary economics. This subsection, together with discussion in the next section, presents some results related to this issue. In the short-run where we have a multitude of equilibria, the analysis on monetary policies is meaningless due to the indeterminacy of outcomes unless we impose some restrictions on the behavior of the agents in the private sector, i.e., how agents react to a change in monetary policies. We study the model under two different scenarios.

Consider first the scenario in which a change in money supply does not cause any reaction of the agents, i.e., their strategies remain the same even after M changes. To make the analysis simple, consider the canonical p-SPE for some p, and assume that the government increases the money supply from M to M' by giving p units of fiat money each to some of the non-money holders. This increases the fraction of money holders from M/p to M'/p. Since they follow the strategies prescribed in the canonical p-SPE, the price remains unchanged, and therefore, the real stock of money is also increased to $m' = M'/p$. The welfare changes from $\min\{mu, (1-m)u\}$ to $\min\{m'u, (1-m')u\}$. Thus, the monetary policy is effective. An increase in

[16]See footnote 15.

the real stock of money increases the welfare up to $m = 1/2$ and decreases it beyond $m = 1/2$.

Next, if the agents adjust their strategies to keep the real stock of money constant, then a change in money supply is completely neutralized unless it triggers a disequilibrium adjustment process. To see this point, suppose that the government gives $p\frac{M'-M}{M}$ units of money to the money holders. Then all the money holders have $p' = pM'/M$ units of money. If all the agents notice it and change their strategies to $\sigma_{p'}$, the strategy used in the canonical p'-SPE, then no real variable is affected, and the monetary policy is completely ineffective.

In the present framework, there is no force that makes prices, or the strategies of the agents, react to a change in money supply, provided that we stick to the equilibrium analysis. If we consider the process of price adjustment, we have a totally different story. In order to express it, we now turn to an evolutionary approach.

4 Evolutionary Stability

4.1 Stability Concept

This section examines the evolutionary stability of stationary distributions. In order for a distribution to be evolutionarily stable, we require that the original population fare at least as good as any small group of mutants in the long-run provided that the agents are sufficiently patient. The formal definition is given below.

Definition 3 A robust equilibrium μ is said to be *evolutionarily stable* if for all $\gamma > 0$, there exists $\bar{\delta} \in (0,1)$ such that for all $\delta \in (\bar{\delta}, 1)$, there exists $\bar{\epsilon} > 0$ such that for all $\epsilon \in (0, \bar{\epsilon})$,

$$V\left(\mu, (1-\epsilon)\mu + \epsilon\tilde{\mu}\right) + \gamma > V\left(\tilde{\mu}, (1-\epsilon)\mu + \epsilon\tilde{\mu}\right).$$

for all $\tilde{\mu}$ with $\tilde{\mu}_H = \mu_H$.[17]

This definition is similar to the definition of evolutionarily stable strategy (Maynard Smith and Price [13]). However, the present definition has three

[17] The condition $\tilde{\mu}_H = \mu_H$ is needed to ensure that mutants' money holdings do not increase by mutation.

differences from it. The first, and the most significant, difference is that agents are patient in the present model. Therefore, comparison between the original population and the mutants is made in terms of discounted average payoffs instead of instantaneous payoffs. In calculating these values, it is assumed that the fraction of the mutants remains "small". Incorporating a possibility of growing population of mutants complicates the analysis, which we do not deal with in the present paper.

Second, the term γ makes the concept weaker than otherwise. Due to this term, the original population survives even if it is "a little" worse than the mutants. In fact, in the present definition, mutants cannot invade the population unless they fare better than the original population in the long-run. This reflects the idea that a "small" one-shot gain is considered negligible, and only constant gains over time would be counted as a threat to the original population. Our claim in the subsequent subsection is that most of equilibria are invaded by mutants even with such a weak concept.

Finally, the mutant population may include "dummies", i.e., those who do not actually mutate. This way we save some cumbersome notation.

4.2 Result

This subsection shows that the class of the efficient single price equilibria is the only evolutionarily stable distributions. To begin with, the following theorem shows that any inefficient equilibrium cannot be evolutionarily stable.

Theorem 5 Every evolutionarily stable equilibrium is efficient.

For the detail of the proof, see Appendix C.

To see the intuition behind the result, consider the following example. Suppose that the economy is trapped in the canonical $4M$-SPE, μ_{4M}. Then there is an excess supply due to a high price. Also note that only market places $1, \ldots, K$ are used. Consider now the mutants who visit only market places unused by the original population. In other words, suppose that a small fraction ϵ of money holders mutate at the beginning of period 1 to take strategy $\tilde{\sigma} = (\tilde{\lambda}, \tilde{o}, \tilde{\beta})$ given by

- $\tilde{\lambda}(\eta) = \begin{cases} (K+k, B) & \text{if } \eta \geq 2M, \\ (K+k+1, A) & \text{otherwise,} \end{cases}$

- $\tilde{o}(\eta) = \begin{cases} 2M & \text{if } \eta < 2M, \\ \infty & \text{otherwise,} \end{cases}$

- $\tilde{\beta}(\eta) = \begin{cases} 2M & \text{if } \eta \geq 2M, \\ \eta & \text{otherwise.} \end{cases}$

The mutant distribution is denoted by $\tilde{\mu}$. It is given by

$$\tilde{\mu}(4M; \tilde{\sigma}) = \frac{1}{4},$$
$$\tilde{\mu}(0; \tilde{\sigma}) = \frac{3}{4}.$$

Let $\hat{\mu}^1 = (1-\epsilon)\mu_{4M} + \epsilon\tilde{\mu}$ be the entire distribution after the mutation at period 1. Since the mutants do not interact with the agents in the original population, the distribution of the money holdings of the mutants evolves independently of the original population. Note that the distribution of the original population remains unchanged. On the other hand, the mutant distribution at period 2 changes from $\tilde{\mu}$ to $\tilde{\mu}^2$, which is given by

$$\tilde{\mu}^2(2M; \tilde{\sigma}) = \frac{1}{2},$$
$$\tilde{\mu}^2(0; \tilde{\sigma}) = \frac{1}{2}.$$

It remains $\tilde{\mu}^2$ thereafter. The entire distribution after period 2 is given by

$$\hat{\mu}^t = (1-\epsilon)\mu_{4M} + \epsilon\tilde{\mu}^2 \quad t \geq 2.$$

Thus, we obtain

$$V(\mu_{4M}, \hat{\mu}) = \frac{1}{4}u,$$

and

$$V(\tilde{\mu}, \hat{\mu}) = \frac{1+\delta}{4}u,$$

which implies $V(\tilde{\mu}, (1-\epsilon)\mu_{4M} + \epsilon\tilde{\mu}) > V(\mu_{4M}, (1-\epsilon)\mu_{4M} + \epsilon\tilde{\mu}) + \gamma$ for a sufficiently small γ and a sufficiently large δ. Hence, the canonical $4M$-SPE is not immune to the invasion of these mutants.

In the above example, the mutants use empty market places to start an efficient trading pattern. The same logic applies to an inefficient equilibrium in which all market places are visited by some agents. The only modification is that instead of visiting empty market places, mutants visit thin market places, which are used by sufficiently small fraction of agents. Migrating to such places, the mutants can make the influence from the original population arbitrarily small and attain a "near" efficient value.

Theorem 5 together with Theorem 4 implies the following corollary. It states that any evolutionarily stable outcome is not only efficient but also a $2M$-SPE.

Corollary 1 If μ is evolutionarily stable, then μ is a $2M$-SPE.

Our next result states that the canonical $2M$-SPE is evolutionarily stable. Also, this result implies the existence of evolutionarily stable equilibrium.

Theorem 6 The canonical $2M$-SPE is evolutionarily stable.

For the detail of the proof, see Appendix D.

Here, we draw a sketch of the proof. Note first that until some agents mutate, agents in the original population maintain an efficient trading pattern. They maintain the pattern even after some mutation occurs until they are "infected" by the mutants either by direct contact or by indirect contagion. If the fraction of mutants is sufficiently small, a sufficiently large fraction of the original population can keep the original efficient trading pattern for a sufficiently long time. In other words, the original population can get a payoff sufficiently close to $u/2$. On the other hand, the mutants can obtain at most "a little" above $u/2$ no matter how well they behave. Therefore, mutants cannot invade the original distribution.

By Corollary 1 and Theorem 6, we conclude that the class of $2M$-SPE, which prescribes almost all agents to alternate between production and consumption with a trading price of $2M$, is the essentially unique evolutionarily stable distributions.[18]

5 Multiple Currencies

This section introduces an additional medium of exchange. Its main purpose is to show that the present model has a potential of analyzing some issues

[18]We cannot say it is the unique distribution since we can replicate the canonical $2M$-SPE by distributing the population to other market places.

on multiple currencies. Therefore, the analysis of this section is illustrative rather than comprehensive.

Suppose that there is another fiat money, called good 0^*, which is of the same nature as, but distinguishable from, good 0. Although this economy also has a multitude of equilibria, we characterize only one class and analyze the effects of changes in money supply. The class of equilibria we focus on is the one in which both goods 0 and 0^* are used in different market places. One example is the situation in which two currencies coexist such as a dual currency economy where dollar is used for specific transactions, while domestic currency is used for other transactions.

Let M and M^* be the nominal stocks of fiat monies 0 and 0^*, respectively. Consider the following canonical distribution:

- $\mu(0;\sigma) = n - m$,
- $\mu(p;\sigma) = m$,
- $\mu(0;\sigma^*) = n^* - m^*$,
- $\mu(p^*;\sigma^*) = m^*$,

where n and $n^*(= 1 - n)$ are the fractions of the agents who take σ and σ^*, respectively, and $m = \frac{M}{p}$ and $m^* = \frac{M^*}{p^*}$ are the total "real" stocks of fiat monies 0 and 0^*, respectively.

We let fiat money 0 be used in market places $1, \ldots, K$, while fiat money 0^* in $K+1, \ldots, 2K$. The strategy $\sigma = (\lambda, o, \beta)$ is the same as σ_p defined in the canonical p-SPE whenever an agent does not hold 0^*. To be precise, it is defined as follows:

- $\lambda(\eta, \eta^*) = \begin{cases} (k, B) & \text{if } \eta \geq p, \\ (K+k, B) & \text{if } \eta < p \text{ and } \eta^* \geq p^*, \\ (k+1, A) & \text{otherwise,} \end{cases}$
- $o(\eta, \eta^*) = \begin{cases} p & \text{if } \eta < p \text{ and } \eta^* < p^*, \\ \infty & \text{otherwise,} \end{cases}$
- $\beta(\eta, \eta^*) = \begin{cases} p & \text{if } \eta \geq p, \\ p^* & \text{if } \eta < p \text{ and } \eta^* \geq p^*, \\ \eta & \text{otherwise.} \end{cases}$

Similarily, $\sigma^* = (\lambda^*, o^*, \beta^*)$ is defined as follows:

- $\lambda^*(\eta, \eta^*) = \begin{cases} (K+k, B) & \text{if } \eta^* \geq p^*, \\ (k, B) & \text{if } \eta^* < p^* \text{ and } \eta \geq p, \\ (K+k+1, A) & \text{otherwise,} \end{cases}$

- $o(\eta, \eta^*) = \begin{cases} p^* & \text{if } \eta^* < p^* \text{ and } \eta < p, \\ \infty & \text{otherwise,} \end{cases}$

- $\beta(\eta, \eta^*) = \begin{cases} p^* & \text{if } \eta^* \geq p^*, \\ p & \text{if } \eta^* < p^* \text{ and } \eta \geq p, \\ \eta^* & \text{otherwise,} \end{cases}$

To preclude degenerate cases from the analysis, we assume that m, m^*, $n-m$, and n^*-m^* are all strictly positive.

Theorem 7 For all p and p^* satisfying $\frac{M}{p} + \frac{M^*}{p^*} < 1$, and all $\delta \in (0,1)$, the canonical distribution μ is a stationary equilibrium if and only if either $\frac{m}{n} = \frac{m^*}{n^*}$ or $\frac{n-m}{n} = \frac{m^*}{n^*}$ (or both) holds.

Proof: The only incentive constraint we have to check is whether or not the agents are willing to switch from one currency to the other for their transactions. The rest of the proof is omitted since it is essentially the same as that of Theorem 1. Furthermore, for the *if*-part, we check only the case of $\frac{m}{n} = \frac{m^*}{n^*} > \frac{1}{2}$. Other cases can be proven in a similar manner.

Let $r = \frac{n-m}{m}$ and $r^* = \frac{n^*-m^*}{m^*}$. Then the value functions are written as[19]

$$V(\sigma, \ell p; \mu) = r\left(1 - \left[\frac{\delta r}{1-\delta+\delta r}\right]^{\ell-1} \frac{\delta r}{1+\delta r}\right) u, \quad \ell \geq 0,$$

and

$$V(\sigma^*, \ell p^*; \mu) = r^*\left(1 - \left[\frac{\delta r^*}{1-\delta+\delta r^*}\right]^{\ell-1} \frac{\delta r^*}{1+\delta r^*}\right) u, \quad \ell \geq 0.$$

[19]In this proof, we denote $V(\sigma, \eta; \mu)$ and $V(\sigma^*, \eta^*; \mu)$ for $V(\sigma, \eta, 0; \mu)$ and $V(\sigma^*, 0, \eta^*; \mu)$, respectively.

Since $r = r^*$ holds, $V(\sigma, \ell p; \mu) = V(\sigma^*, \ell p^*; \mu)$ holds, too. Therefore, a non-money holder is indifferent between the two currencies. Money holders are strictly better off using what they possess before earning additional units of money. Therefore, no agent has an incentive to deviate.

To show the *only-if*-part, suppose the contrary, i.e., that neither $\frac{m}{n} = \frac{m^*}{n^*}$ nor $\frac{n-m}{n} = \frac{m^*}{n^*}$ holds. We show the case of $\frac{m}{n} < \frac{m^*}{n^*} < \frac{1}{2}$ only. Other cases are proven in a similar manner. In this case, a non-money holder taking σ has an incentive to switch to σ^*. For if he follows σ, his value is

$$V_0 = \left(1 - \frac{1}{1 + \delta \frac{m}{n-m}}\right) u,$$

while if he took σ^*, his value would be

$$V_0^* = \left(1 - \frac{1}{1 + \delta \frac{m^*}{n^* - m^*}}\right) u,$$

and therefore, $\frac{m}{n} < \frac{m^*}{n^*} < \frac{1}{2}$ implies $V_0^* > V_0$. ∎

Next, we analyze the effects of changes in money supply. Suppose that the money supply of one currency, say, 0 changes. Then it may lead to changes in two types of variables, price levels and the fraction of the users of the two fiat money. In particular, a switch from one currency to the other necessarily occurs if the price adjustment is not swift since one currency is more attractive than the other until the price adjustment is completed. Therefore, it is worthwhile to see how the fraction n of the agents using 0 is adjusted, keeping the price levels constant.

Consider the case of $\frac{m}{n} = \frac{m^*}{n^*} < \frac{1}{2}$ first. Suppose that the issuer of fiat money 0 increases its money supply a little to $M' > M$ so that $\frac{m^*}{n^*} < \frac{m'}{n} < \frac{1}{2}$ holds where $m' = M'/p$. Then fiat money 0 becomes more attractive than 0^* as shown in the proof of Theorem 7. As a result, non-money holders taking σ start switching from 0^* to 0. The fraction n increases until the ratios of money holders to non-money holders become equal between the two currencies. This adjustment process may lead to a new equilibrium. In this equilibrium, the level of welfare is higher than before since the new ratio, denoted m'/n', is closer to the optimal ratio, 1/2, than the old one.

Consider next the case of $\frac{m}{n} = \frac{m^*}{n^*} > \frac{1}{2}$. In this case, the sum of the real stocks of the two currencies is beyond the optimal level. Therefore, it may be beneficial for the money issuer to reduce its money supply. Suppose the issuer of fiat money 0 reduces it. Then fiat money 0 becomes more attractive than 0^*, and n increases as in the previous case. However, the more n becomes, the more attractive 0 becomes than 0^*. In particular, when $n > n^*$, this process may continue until no agent uses 0^*.[20] Over-issuance makes the currency itself vulnerable. Therefore, the money issuers may have an incentive to restrain themselves from collecting seigniorage too much. In the countries which experience hyper-inflation, monopolizing the sole medium of exchange is detrimental to the economy since this self-discipline works only when there is a competitor.

The above result implies that competition between currencies imposes discipline on the money issuers as argued by Hayek [8]. On the other hand, if, by law, agents have to accept either money at the fixed rate, then the above adjustment mechanism does not work, and we will have the problem of over-issuance by the issuers who try to collect seigniorage. It is important that people can choose which currency to use.

6 Concluding Remarks

We have analyzed a search theoretic model of money with market places. We have adopted two solution concepts in the main analysis, stationary equilibrium and evolutionarily stable distribution.

We have viewed the equilibrium approach as a proxy for the short-run situation. In some equilibria, one market place is used for all transactions, while in others, markets are specialized; at most one commodity good is traded in each market place. There are a continuum of equilibria with different price and welfare levels. There also exist dual price equilibria, in which the same good is traded at two different prices. We have also analyzed some effects of monetary policies. Money supply can be changed without causing a change in price level if agents do not alter their strategies. It corresponds to the case of price rigidity, which is sometimes considered as a ground for Keyensian economics. In such a case, the monetary policy is effective in that it reduces, say, excess demand. On the other hand, if agents swiftly adjust their behavior to the real stock of money, the monetary policy is ineffective.

[20] When n is very small, this process may converge to an asymmetric equilibrium since an increase in n has a large effect on $\frac{m}{n}$.

We have adopted the evolutionary approach as a proxy for the long-run situation. In the long-run, those who fare better than others survive, while those who do not do well shift from one strategy to a better one, or disappear from the market. As a result, an efficient single price equilibrium prevails. This evolutionary view of the markets is found in, among others, Alchian [1] and Friedman [4].[21]

When destroying some equilibrium by mutation, mutants utilize some inactive places to establish a new transaction pattern. This idea of using inactive places for deviation is similar to the ideas of the secret hand shake in Robson [17] and the cheap-talk in Matsui [12]. In Matsui [12], an unused message is sent to signal others that one is a new type. They take a more efficient strategy profile than before only if they both send this new message. In a similar manner, successful mutants of the present model choose inactive market places to establish a more efficient trading pattern than before.[22]

To conclude the paper, four remarks are in order. First, the restriction to Markov perfect equilibria plays a crucial role in the proof of the essential uniqueness result of the evolutionarily stable outcome. For example, Markov perfection excludes the possibility of punishment against mutants, e.g., sellers' price cut. Without the restriction, agents may take a strategy according to which they trail the mutants who go to inactive places and behave as sellers so that the mutants cannot increase the probability of matching with buyers even if they cut the price. We do not think that this change in the results undermines our analysis. Rather, the lack of retaliation and punishment against price cut is an essential feature that makes the price adjustment mechanism work, and the concept of Markov perfection expresses it in a simple form.

Second, the matching technology of the present model exhibits constant returns to scale, i.e., the matching probability depends only on the relative size of agents visiting side A and those visiting side B. Although it serves a benchmark, one may wonder how the results would change if the matching technology is that of increasing returns to scale, i.e., the larger the absolute

[21] It should be noted that the assumption of simultaneous trials is made for the sake of analytical simplicity rather than realistic description of changes in behavior. In reality, even if a seller cuts its price, she has to wait for a while to attract new customers. People gradually realize that there is the new seller who sells goods cheaper than other stores. Effectively speaking, when a seller tries an inactive market place with a new price, buyers need to visit the seller not necessarily right after the seller's trial, but only before it disappears from the market. The effect of a price cut will appear only gradually.

[22] See also Ely [3] and Mailath, Samuelson, and Shaked [11].

size of a market place, the greater is the matching probability. In such a case, some results, especially, the one in Section 4 on evolution may be modified since creating a small new market may not pay off. Indeed, such a new market never appears if the degree of increasing returns to scale is too large as in Iwai [9], in which the probability of matching goes to zero as the size of the market tends to zero. If, on the other hand, the scale economy exists but not too large, then the further the price is away from $2M$, the more likely is the corresponding equilibrium to be destroyed by the mutants creating a new transaction pattern since a gain from a better seller-buyer ratio exceeds a loss caused by the effect of scale economy.

Third, we have analyzed the effects of evolutionary pressure without any specification of explicit dynamics. Although we have obtained the efficiency result, analyses with explicit dynamics would deepen our understanding of the process of price adjustment.

Fourth, Section 5 has extended our model to an economy with multiple currencies. To further study such an economy, we may introduce issuers explicitly, examining their incentives to issue fiat money, and the way they interact with each other. We leave these studies for the future research.

References

[1] Armen A. Alchain. Uncertainty, evolution, and economic theory. *Journal of Political Economy*, 58:211–221, 1950.

[2] Dean Corbae, Ted Temzelides, and Randall Wright. Matching and money. 2000.

[3] Jeffrey Ely. Local conventions. 1995.

[4] Milton Friedman. The methodology of positive economics. In *Essays in Positive Economics*, chapter 1. University of Chicago Press, 1953.

[5] David Gale and Lloyd Shapley. College admissions and the stability of marriage. *American Mathematical Monthly*, 69:9–15, 1962.

[6] Edward J. Green and Ruilin Zhou. A rudimentary random-matching model with divisible money and prices. *Journal of Economic Theory*, 81:252–271, 1998.

[7] Fumio Hayashi and Akihiko Matsui. A model of fiat money and barter. *Journal of Economic Theory*, 68:111–132, 1996.

[8] Friedrich A. Hayek. *Denationalisation of Money: An Analysis of the Theory and Practice of Concurrent Currencies*. London: Institute of Economic Affairs, 1976.

[9] Katsuhito Iwai. The bootstrap theory of money: A search-theoric foundation of monetary economics. *Structural Change and Economic Dynamics*, 7:451–477, 1996.

[10] Nobuhiro Kiyotaki and Randall Wright. On money as a medium of exchange. *Journal of Political Economy*, 97(4):927–954, 1989.

[11] George J. Mailath, Larry Samuelson, and Avner Shaked. Endogenous interactions. 1997.

[12] Akihiko Matsui. Cheap talk and cooperation in a society. *Journal of Economic Theory*, 54:245–58, 1991.

[13] John Maynard Smith and G. R. Price. The logic of animal conflict. *Nature*, 246:15–18.

[14] Espen R. Moen. Competitive search equilibrium. *Journal of Political Economy*, 105(2):385–411, 1997.

[15] James D. Montgomery. Equilibrium wage dispersion and interindustry wage differentials. *Quarterly Journal of Economics*, 106:163–179, 1991.

[16] Michael Peters. Ex ante price offers in matching games non-steady states. *Econometrica*, 59(5):1425–1454, 1991.

[17] Arthur J. Robson. Efficiency in evolutionary games: Darwin, nash, and the secret handhake. *Journal of Theoretical Biology*, 144:379–96, 1990.

[18] Lloyd S. Shapley and Martin Shubik. Trade using one commodity as a means of payment. *Journal of Political Economy*, 5(5):937–68, 1977.

[19] Shouyong Shi. A divisible search model of fiat money. *Econometrica*, 65(1):75–102, 1997.

[20] Alberto Trejos and Randall Wright. Search, bargaining, money, and prices. *Journal of Political Economy*, 103(1):118–141, 1995.

Appendices

A Incentive Compatibility Conditions in the Canonical DPE

We show that for all δ, there exists an integer ℓ^* ($1 \leq \ell^* < n$) such that σ_1 and σ_n are the best responses to $\mu_{(p,np)}$ whenever $r = 1/n$. The proof is divided into two parts.

Part 1: First, we determine ℓ^*. For this purpose, we introduce the following auxiliary strategies, value functions, and conditions.

Let $\tilde{r} = h_{01}/h_1$. Then let

$$\begin{aligned}
&\tilde{V}_0 = \delta \tilde{V}_1, \text{ and} \\
&\tilde{V}_\ell = \tilde{r}\Big((1-\delta)u + \delta \tilde{V}_{\ell-1}\Big) + (1-\tilde{r})\delta \tilde{V}_\ell, \quad \ell \geq 1.
\end{aligned}$$

This is an auxiliary value function. It corresponds to the auxiliary strategy according to which an agent uses the low-price markets only, irrespective of the agent's money holdings. Also, let $\hat{r} = h_n/h_{0n}$. Then let

$$\begin{aligned}
&\hat{V}_0 = \hat{r}\delta \hat{V}_n + (1-\hat{r})\delta \hat{V}_0, \\
&\hat{V}_\ell = \delta^{n-\ell}\hat{V}_n, \quad 1 \leq \ell < n, \text{ and} \\
&\hat{V}_n = (1-\delta)u + \delta \hat{V}_0.
\end{aligned}$$

This is another auxiliary value function. It corresponds to another auxiliary strategy. If one has money holdings less than p, he goes to the high-price market as a seller. If he has money holdings greater than or equal to p but less than np, he goes to the low-price market as a seller. If he has money holdings greater than or equal to np, he goes to the high-price market as a buyer.

We choose ℓ^*, $\tilde{r}$, and $\hat{r}$ so that the following conditions are satisfied.

[**C0**] $\tilde{V}_0 = \hat{V}_0$.

[**C1**] $\tilde{V}_\ell \geq \hat{V}_\ell$ if $1 \leq \ell \leq \ell^*$.

[**C2**] $\tilde{V}_\ell \leq \hat{V}_\ell$ if $\ell^* < \ell \leq n$.

From the description of $\tilde{V}$'s and [C1], we obtain

$$\tilde{V}_0 = \frac{\delta \tilde{r}}{1+\delta\tilde{r}} u = \frac{\delta \hat{r}}{1+\delta\hat{r}} u = \hat{V}_0, \tag{1}$$

Then we obtain $\tilde{r} = \hat{r}$. From now on, this common ratio is denoted by r. Then sequentially applying (1) to $\tilde{V}$'s, we obtain

$$\tilde{V}_\ell = ru - \left[\frac{\delta r}{1-\delta+\delta r}\right]^{\ell-1} \frac{\delta r}{1+\delta r} ru, \quad \ell \geq 1. \tag{2}$$

Similarly, using the description of $\hat{V}$'s and (1), we obtain

$$\hat{V}_\ell = \delta^{n-\ell} \frac{1-\delta+\delta r}{1+\delta r} u, \quad 1 \leq \ell \leq n. \tag{3}$$

It is verified that $\tilde{V}_\ell$ is concave in ℓ, and that $\hat{V}_\ell$ is convex in ℓ. Therefore, if we prove that $\tilde{V}_1 \geq \hat{V}_1$ and $\tilde{V}_n \leq \hat{V}_n$ hold for some r, then there exists ℓ^* between 1 and n such that [C1] and [C2] hold. Indeed, it is verified that $\tilde{V}_1 \geq \hat{V}_1$ holds if $r \geq 1/n$. After tedious calculation, it is also verified that $\tilde{V}_n \leq \hat{V}_n$ holds if $r = 1/n$. Thus, if $r = 1/n$, then there exists ℓ^* between 1 and n such that [C1] and [C2] hold. Moreover, if δ is close to one, then $\tilde{V}_1 \geq \hat{V}_1$ is approximately equivalent to $r \geq 1/n$, while $\tilde{V}_n \leq \hat{V}_n$ is approximately equivalent to $r \leq 1/n$ (still, both inequalities hold if $r = 1/n$). Thus, in the limit of δ going to one, $r = 1/n$ necessarily holds. ∎

Part 2: Next, in the case of $r = 1/n$, we show that all the incentive compatibility conditions hold whenever [C0],[C1], and [C2] hold.

Using $\tilde{V}$'s and $\hat{V}$'s, we can write the value function in the canonical DPE as follows:

$$V_\ell^* = \begin{cases} \tilde{V}_0 = \hat{V}_0 & \text{if } \ell = 0, \\ \tilde{V}_\ell & \text{if } 1 \leq \ell \leq \ell^*, \\ \hat{V}_\ell & \text{if } \ell^* < \ell \leq n, \\ (1-\delta)u + \delta V_{\ell-n}^* & \text{if } \ell > n. \end{cases}$$

The incentive compatibility conditions that we need to verify are the following ones:

- With regard to non-money holders:

C0' Non-money holders are indifferent between selling at the low-price market and selling at the high-price market.

- With regard to money holders with ℓp $(0 < \ell \leq \ell^*)$:

  C1SR they never want to sell at the high-price market.

  C1SP they never want to sell at the low-price market.

- With regard to money holders with ℓp $(\ell^* < \ell \leq n)$:

  C2SR they never want to sell at the high-price market.

  C2BP they never want to buy at the low-price market.

- With regard to money holders with ℓp $(\ell \geq n)$:

  C3SR they never want to sell at the high-price market.

  C3SP they never want to sell at the low-price market.

  C3BP they never want to buy at the low-price market.

To start with, we prove the following fact:

Fact 1 $\delta^{n-\ell^*-1} - r\left(1 + \delta + \ldots + \delta^{n-\ell^*-1}\right) \geq 0.$

Proof of Fact 1: By [C1], we have $\tilde{V}_{\ell^*} \geq \hat{V}_{\ell^*}$, while, by [C2], we have $\tilde{V}_{\ell^*+1} \leq \hat{V}_{\ell^*+1}$. Then we have

$$\frac{\delta r}{1-\delta+\delta r}\left(\tilde{V}_{\ell^*} - \hat{V}_{\ell^*}\right) + \hat{V}_{\ell^*+1} - \tilde{V}_{\ell^*+1} \geq 0.$$

It follows that we have

$$\delta^{n-\ell^*-1} - r\left(1 + \delta + \ldots + \delta^{n-\ell^*-1}\right) \geq 0.$$

∎

C0' : This condition is equivalent to [C0].

C1SR : This condition is given by

$$V^*_\ell \geq r\delta V^*_{\ell+n} + (1-r)\,\delta V^*_\ell \quad 0 < \ell \leq \ell^*,$$

which is equivalent to

$$(1-\delta)\left((1-\delta+\delta r)\, r-\left(\frac{\delta r}{1-\delta+\delta r}\right)^{\ell-1}\delta r^{2}\right) u \geq 0. \tag{4}$$

Since (4) holds if $\ell=1$, and the left hand side of (4) is increasing in ℓ, it always holds.

C1SP : This condition is given by

$$V_{\ell}^{*} \geq \delta V_{\ell+1}^{*} \quad 0<\ell \leq \ell^{*}. \tag{5}$$

- If $\ell<\ell^{*}$, (5) is equivalent to

  $$(1-\delta)\, r\left(1-\left(\frac{\delta r}{1-\delta+\delta r}\right)^{\ell}\right) u \geq 0,$$

  which always holds.
- If $\ell=\ell^{*}$, (5) holds since

  $$\begin{aligned} V_{\ell^{*}}^{*} &= \tilde{V}_{\ell^{*}} \\ &\geq \hat{V}_{\ell^{*}} \quad ([\mathrm{C1}]) \\ &= \delta \hat{V}_{\ell^{*}+1} \\ &= \delta V_{\ell^{*}+1}^{*}. \end{aligned}$$

C2SR : This condition is given by

$$V_{\ell}^{*} \geq r \delta V_{\ell+n}^{*}+(1-r)\, \delta V_{\ell}^{*} \quad \ell^{*}<\ell \leq n,$$

which is equivalent

$$\frac{(1-\delta)^{2}}{\delta}\left(\delta^{n-\ell-1}-r\left(1+\ldots+\delta^{n-\ell-1}\right)\right) u \geq 0.$$

This holds since, by Fact 1,

$$\begin{aligned} \left(\delta^{n-\ell-1}-r\left(1+\ldots+\delta^{n-\ell-1}\right)\right) &\geq \delta^{n-\ell-1}-\delta^{n-\ell^{*}-1}+r\left(\delta^{n-\ell}+\ldots+\delta^{n-\ell^{*}-1}\right) \\ &= \delta^{n-\ell-1}\left(1-\delta^{\ell-\ell^{*}}\right)+r\left(\delta^{n-\ell}+\ldots+\delta^{n-\ell^{*}-1}\right) \\ &\geq 0. \end{aligned}$$

C2BP : This condition is given by

$$V_\ell^* \geq r\left((1-\delta)\,u + \delta V_{\ell-1}^*\right) + (1-r)\,\delta V_\ell^* \quad \ell^* < \ell \leq n,$$

which is equivalent to

$$(1-\delta+\delta r)\,V_\ell^* - \delta r V_{\ell-1}^* - (1-\delta)\,ru \geq 0. \tag{6}$$

- If $\ell = \ell^* + 1$, (6) holds since

$$\begin{aligned}\text{l.h.s. of (6)} &= (1-\delta+\delta r)\,\hat{V}_{\ell^*+1} - \delta r \tilde{V}_{\ell^*} - (1-\delta)\,ru \\ &\geq (1-\delta+\delta r)\,\tilde{V}_{\ell^*+1} - \delta r \tilde{V}_{\ell^*} - (1-\delta)\,ru \quad ([\text{C2}]) \\ &= 0.\end{aligned}$$

- If $\ell > \ell^* + 1$, (6) is equivalent to

$$\left(\delta^{n-\ell} - r\left(1 + \ldots + \delta^{n-\ell}\right)\right) u \geq 0.$$

  This holds since, by Fact 1,

$$\begin{aligned}\delta^{n-\ell} - r\left(1 + \ldots + \delta^{n-\ell}\right) &\geq \delta^{n-\ell} - \delta^{n-\ell^*-1} + r\left(\delta^{n-\ell+1} + \ldots + \delta^{n-\ell^*-1}\right) \\ &= \delta^{n-\ell}\left(1 - \delta^{\ell-\ell^*-1}\right) + r\left(\delta^{n-\ell+1} + \ldots + \delta^{n-\ell^*-1}\right) \\ &\geq 0.\end{aligned}$$

C3SR : This condition is given by

$$V_\ell^* \geq r\delta V_{\ell+n}^* + (1-r)\,\delta V_\ell^* \quad \ell \geq n,$$

which is equivalent to

$$(1-\delta)\left((1+\delta r)\,V_\ell^* - \delta r u\right) \geq 0.$$

This holds since, by the monotonicity of V_ℓ,

$$\text{l.h.s.} \geq (1-\delta)\left((1+\delta r)\,V_0 - \delta r u\right) = 0.$$

C3SP : This condition is given by

$$V_\ell^* \geq \delta V_{\ell+1}^* \quad \ell \geq n. \tag{7}$$

We decompose $\ell \geq n$ into $in + j$ where i is an integer greater than or equal to 1, and j is an integer between 0 and $n-1$, then we have

$$V_{in+j}^* = \left(1 - \delta^i\right) u + \delta^i V_j^*. \tag{8}$$

- If $j < n-1$, by (8), (7) is equivalent to

$$(1-\delta)\left(1-\delta^i\right)u + \delta^i\left(V_j^* - \delta V_{j+1}^*\right) \geq 0.$$

This holds since

  - if $j \leq \ell^*$, it is followed by [C1SP],
  - if $j > \ell^*$, it holds that $\hat{V}_j = \delta \hat{V}_{j+1}$.

- If $j = n-1$, by (8), (7) is equivalent to

$$(1-\delta)\left(1-\delta^i\right)u + \delta^i\left(V_{n-1}^* - \delta V_n^*\right) \geq 0.$$

This holds since

  - if $\ell^* < n-1$, it holds $\hat{V}_{n-1} = \delta \hat{V}_n$.
  - if $\ell^* = n-1$, it holds that

$$\begin{aligned}
\text{l.h.s} &= (1-\delta)\left(1-\delta^i\right)u + \delta^i\left(\tilde{V}_{n-1} - \delta \hat{V}_n\right) \\
&\geq (1-\delta)\left(1-\delta^i\right)u + \delta^i\left(\hat{V}_{n-1} - \delta \hat{V}_n\right) \quad \text{([C1])} \\
&= 0.
\end{aligned}$$

C3BP : This condition is given by

$$V_\ell^* \geq r\left((1-\delta)u + \delta V_{\ell-1}^*\right) + (1-r)\delta V_\ell^* \quad \ell \geq n. \tag{9}$$

- If $j = 0$ (recall the decomposition of ℓ), by (8), (9) is equivalent to

$$(1-\delta)\left(1-\delta^{i-1}\right)(1-r)u + \delta^{i-1}\left(V_n^* - r\left((1-\delta)u + \delta V_{n-1}^*\right) - (1-r)\delta V_n^*\right) \geq 0.$$

This holds since

  - if $\ell^* < n-1$, it holds that

$$\begin{aligned}
\text{l.h.s.} &= (1-\delta)\left(1-\delta^{i-1}\right)(1-r)u + \delta^{i-1}(1-\delta)\left((1+\delta r)\hat{V}_n - ru\right) \\
&= (1-\delta)\left(1-\delta^i\right)(1-r)u \\
&\geq 0,
\end{aligned}$$

– if $\ell^* = n-1$, it holds that

$$\begin{aligned}
V_n^* - r\left((1-\delta)u + \delta V_{n-1}^*\right) - (1-r)\delta V_n^* \\
&= (1-\delta+\delta r)\hat{V}_n - r\left((1-\delta)u + \delta\tilde{V}_{n-1}\right) \\
&\geq (1-\delta+\delta r)\tilde{V}_n - r\left((1-\delta)u + \delta\tilde{V}_{n-1}\right) \quad ([\mathrm{C2}]) \\
&= 0.
\end{aligned}$$

- If $j > 0$, by (8), (9) is equivalent to

$$(1-\delta)\left(1-\delta^i\right)(1-r)u + \delta^i\left(V_j^* - r\left((1-\delta)u + \delta V_{j-1}^*\right) - (1-r)\delta V_j^*\right) \geq 0.$$

This holds since

– if $j \leq \ell^*$, it holds that

$$\tilde{V}_j = r\left((1-\delta)u + \delta\tilde{V}_{j-1}\right) - (1-r)\delta\tilde{V}_j,$$

– if $j > \ell^*$, it is derived from [C2BP].

∎

B Proof of Theorem 4

The proof is proceeded by stating the following claims. All the claims are concerned with realizations in any equilibrium that satisfies the assumptions of the theorem.

Claim 1: Almost all the agents either produce or consume in every period.
Proof. Otherwise, some welfare loss is incurred.

Claim 2: In every market place, the fraction of the agents visiting Side A is the same as the fraction of those visiting Side B.
Proof. Otherwise, some fraction of agents fail to be matched, which contradicts Claim 1.

Claim 3: In every market place, each type of good is traded at a single price, if traded at all.

Proof. Suppose the contrary, i.e., that some goods are traded at p and $p' > p$ in some market place. It follows that there are two types of sellers, one offers p, the other offers p'. Moreover, from Claim 1, both sellers should be able to sell goods without fail. Then sellers offering p has an incentive to deviate and offer p'. Repeating this deviation sufficiently many times, the seller can accumulate enough (additional) money to buy an extra good, which is a contradiction.

Claim 4: If a positive fraction of type k agents visit a certain market place, then agents of type other than $k-1$ and $k+1$ (mod K) never visit the other side of this market place. Therefore, each market place must belong to one of the following three categories:

(i) "one-good" market places, i.e., those of which type k agents visit one side, and only $k-1$ type agents or only type $k+1$ (mod K) agents visit the other side;

(ii) "two-good" market places, i.e., those of which type k agents visit one side, and only both type $k-1$ agents and type $k+1$ (mod K) agents visit the other side;

(iii) unused market places, i.e., those places that no agent visits.

Proof. Otherwise, some fraction of agents fail to trade.

Claim 5: There exists no market place in which two types of goods are traded.

We put the proof at the end of this appendix since it is cumbersome.

Claim 6: Every good is traded at the same price.

Proof. From Claim 5, a buyer can buy his consumption good without fail at every market place in which it is sold. Therefore, he has an incentive to buy at a cheaper market place.

Claim 7: The price equals $2M$.

Proof. If the price is strictly greater than $2M$, the fraction of buyers is strictly less than a half, which contradicts Claims 1 and 2.

On the other hand, if the price is strictly less than $2M$, there exist some buyers with money holdings greater than or equal to $2p$ or some sellers with money holdings greater than or equal to p; for if not, some agents hold fiat money they never expect to use and must have already discarded. In order to constantly induce buyers who hold $2p$ units or more, some money holders become sellers. However, the sellers who hold money have an incentive to become buyers since the earlier they consume, the higher is the expected value due to stationarity.

Proof of Claim 5: Suppose the contrary, i.e., that there exists a market place in which two types of goods are traded. We assume, due to symmetry, that market places $1, \ldots, K$ are such places. From Claim 4, we may assume that some of type $k-1$ agents and some of type $k+1$ agents visit Side A of market place k and some type k agents visit Side B of market place k. Let the proportion of type $k-1$ agents in Side A be r $(0 < r < 1)$. We denote the price at which trades are made between type $k-1$ agents and type k agents by p, and the price between type $k+1$ agents and type k agents by p'.

Part 1: We show that $p < p'$. First, $p \leq p'$ holds, for if not, due to symmetry, a type k agent has an incentive to go to market place $k-1$ as a buyer, buying at p', and $k+1$ as a seller, selling at $p > p'$. By this deviation, the agent not only reduces the uncertainty but saves some money (note that the uncertainty comes from the fact that he does not know whether he meets a seller or a buyer). Suppose next that $p = p'$. Then every good is traded at price p in every market place (including the market places in which only one type of good is traded). Some agents of type k with money holdings $\eta' \geq p$ visit Side B of market place k on the equilibrium path. Thus, we have[23]

$$V(\eta') = r\left((1-\delta)u + \delta V(\eta' - p)\right) + (1-r)\,\delta V(\eta' + p).$$

Their incentive compatibility conditions are

$$V(\eta') \geq r\left((1-\delta)u + \delta V(\eta' - p)\right),$$
$$V(\eta') \geq (1-r)\,\delta V(\eta' + p).$$

Then we have

$$(1-\delta)u + \delta V(\eta' - p) = V(\eta') = \delta V(\eta' + p). \tag{10}$$

One of the incentive compatibility conditions for an agent with $\eta' - p$ is

$$V(\eta' - p) \geq \delta V(\eta').$$

[23] We omit σ, μ in the subsequent expressions.

Then we have

$$V(\eta') \geq \frac{1}{1+\delta} u, \tag{11}$$

by (10). On the other hand, an agent with $\eta' + p$ has the following condition:

$$\begin{aligned} V(\eta'+p) &= \frac{1}{\delta} V(\eta') \\ &= \frac{1-\delta^2}{\delta} V(\eta') + \delta V(\eta') \\ &\geq \frac{1-\delta}{\delta} u + \delta V(\eta') \\ &> (1-\delta) u + \delta V(\eta'), \end{aligned}$$

by (10) and (11). Then we have

$$V(\eta'+p) = \delta V(\eta'+2p).$$

Inductively, we obtain

$$V(\eta'+np) = \frac{1}{\delta^n (1+\delta)} u, \quad \forall n \in \mathbb{N}.$$

However, we have

$$\lim_{n\to\infty} V(\eta'+np) = \infty,$$

which contradicts

$$V(\eta) \leq u \quad \forall \eta \in \mathbb{R}_+.$$

Part 2: We choose $\bar{p} \in \mathbb{R}_+$, $m, n \in \mathbb{N}$ such that $m\bar{p} = p$, $n\bar{p} = p'$, and m and n have no common divisor but 1 if we can find such numbers; otherwise, we choose $\bar{p}, \varepsilon \in \mathbb{R}_+$, $m, n \in \mathbb{N}$ such that $p = m\bar{p}$, $p' = n\bar{p} + \varepsilon$, m and n have no common divisor but 1, and ε is small enough.

Given an $\bar{\eta}$, we define the following Markov strategy $\sigma = (\lambda, o, \beta)$.

- $\lambda(\eta) = \begin{cases} (k-1, A) & \text{if } \eta \geq \bar{\eta} + \bar{\ell}\bar{p}, \\ (k, B) & \text{if } \bar{\eta} + \bar{\ell}\bar{p} > \eta \geq \bar{\eta} + m\bar{p}, \\ (k+1, A) & \text{if } \eta < \bar{\eta} + m\bar{p}, \end{cases}$

- $o(\eta) = \begin{cases} n\bar{p} & \text{if } \bar{\eta} + \bar{\ell}\bar{p} > \eta \geq \bar{\eta} + m\bar{p}, \\ p & \text{if } \eta < \bar{\eta} + m\bar{p}, \\ \infty & \text{otherwise}, \end{cases}$

- $\beta(\eta) = \begin{cases} p' & \text{if } \eta \geq \bar{\eta} + \bar{\ell}\bar{p}, \\ p & \text{if } \bar{\eta} + \bar{\ell}\bar{p} > \eta \geq \bar{\eta} + m\bar{p}, \\ \eta & \text{if } \eta < \bar{\eta} + m\bar{p}. \end{cases}$

where $\bar{\ell} = 3mn + m + n$. There is room for arbitrage, and this strategy enables an agent to buy at the lower price p and sell at the higher price p'. We show that every agent can obtain more than $\frac{1}{2}u$ by taking σ, which will be a contradiction.

Suppose that an agent with money holdings $\eta \in [\bar{\eta}, \bar{\eta} + \bar{\ell}\bar{p} + n\bar{p})$ takes the above strategy. Then we construct a probability space $\mathcal{P}^{\eta}$ which governs the stochastic process of this agent's money holdings. Moreover, this stochastic process is a time-homogeneous Markov process with the state space $[\bar{\eta}, \bar{\eta} + \bar{\ell}\bar{p} + n\bar{p})$. We let $L \stackrel{\text{def}}{=} [\bar{\eta}, \bar{\eta} + m\bar{p})$ and $H \stackrel{\text{def}}{=} [\bar{\eta} + \bar{\ell}\bar{p}, \bar{\eta} + \bar{\ell}\bar{p} + n\bar{p})$.

We define various stochastic processes. First, a stochastic process $(S^t)_{t\geq 0}$ on $\mathcal{P}^{\bar{\eta}}$ is recursively defined as follows:

- $S^0 = 0$.
- For $t \geq 1$,

$$S^t = \begin{cases} 0 & \text{if } S^{t-1} = 1, \eta_t \in L, \text{ or } S^{t-1} = 0, \eta_t \notin H, \\ 1 & \text{if } S^{t-1} = 1, \eta_t \notin L, \text{ or } S^{t-1} = 0, \eta_t \in H. \end{cases}$$

This stochastic process switches to 1 when η enters the region of H, and to 0 when it enters L.

Next, stochastic processes $(N_{LH}^t)_{t\geq 0}$ and $(N_{HL}^t)_{t\geq 0}$ on $\mathcal{P}^{\bar{\eta}}$ are recursively defined as follows:

- $N_{LH}^0 = N_{HL}^0 = 0$.

- For $t \geq 1$,

$$N^t_{LH} = \begin{cases} N^{t-1}_{LH} + 1 & \text{if } S^{t-1} = 0, \eta_t \in H, \\ N^{t-1}_{LH} & \text{otherwise.} \end{cases}$$

$$N^t_{HL} = \begin{cases} N^{t-1}_{HL} + 1 & \text{if } S^{t-1} = 1, \eta_t \in L, \\ N^{t-1}_{HL} & \text{otherwise.} \end{cases}$$

(N^t_{LH}) (resp. (N^t_{HL})) counts the number of times that η moves from L to H, (resp. H to L).

Finally, stochastic processes $(N^t_B)_{t\geq 0}$ and $(N^t_S)_{t\geq 0}$ on $\mathcal{P}^{\bar{\eta}}$ are recursively defined as follows:

- $N^0_B = N^0_S = 0$.
- For $t \geq 1$,

$$N^t_B = \begin{cases} N^{t-1}_B + 1 & \text{if he buys his consumption goods at period } t, \\ N^{t-1}_B & \text{otherwise,} \end{cases}$$

$$N^t_S = \begin{cases} N^{t-1}_S + 1 & \text{if he sells his production goods at period } t, \\ N^{t-1}_S & \text{otherwise.} \end{cases}$$

(N^t_B) (resp. (N^t_S)) counts the number of the periods at which the agent acts as a buyer (resp. seller).

Lemma 1 There exists a positive constant N such that

$$\lim_{t\to\infty} \frac{N^t_{LH}}{t} = \lim_{t\to\infty} \frac{N^t_{HL}}{t} \geq N \text{ a.s.}$$

Proof of Lemma 1: We define random variables τ^η's on $\mathcal{P}^\eta$ as follows:

$$[\tau^\eta = t] \stackrel{\text{def}}{=} \cup_{\bar{t}=1}^{t-1} [\eta_1, \ldots, \eta_{\bar{t}-1} \notin L, \eta_{\bar{t}} \in L, \eta_{\bar{t}+1}, \ldots, \eta_{t-1} \notin H, \eta_t \in H] .$$

Moreover, we define D, P as[24]

$$D = 2 + \lfloor \frac{\bar{\ell} - m}{m} \rfloor + \lfloor \frac{\bar{\ell} - m}{n} \rfloor.$$
$$P = r^{\lfloor \frac{\bar{\ell}-m}{m} \rfloor} \times (1-r)^{\lfloor \frac{\bar{\ell}-m}{n} \rfloor} .$$

[24] $\lfloor x \rfloor$ is the least integer more than or equal to x.

Then we have

$$\sup_{\eta\in[\bar{\eta},\bar{\eta}+\bar{\ell}\bar{p}+n\bar{p})} E^{\eta}\left[\tau^{\eta}\right] \leq \sum_{i=1}^{\infty} iD\left(1-P\right)^{i-1}P$$
$$= \frac{D}{P}.$$

where E^{η} is the expectation operator under $\mathcal{P}^{\eta}$. By a Markov property, we have

$$\lim_{t\to\infty}\frac{N_{LH}^{t}}{t} \geq \frac{1}{\sup E^{\eta}\left[\tau^{\eta}\right]} \quad \text{a.s.}$$
$$\geq \frac{P}{D} \quad \text{a.s.}$$
$$> 0 \quad \text{a.s.}$$

On the other hand, we have

$$N_{LH}^{t} \leq N_{HL}^{t} \leq N_{LH}^{t} + 1.$$

Combining these inequalities, we obtain

$$\lim_{t\to\infty}\frac{N_{HL}^{t}}{t} = \lim_{t\to\infty}\frac{N_{LH}^{t}}{t} \geq \frac{P}{D} > 0 \quad \text{a.s.}$$

Let $N = P/D$, and this completes the proof of the lemma. ∎

Using Lemma 1, we have

$$N_{B}^{t} - N_{S}^{t} \geq \left(1 + \lfloor\frac{\bar{\ell}-n-m}{m}\rfloor\right)N_{HL}^{t} - \left(1 + \lfloor\frac{\bar{\ell}+n-m}{n}\rfloor\right)N_{LH}^{t}$$
$$= \left(3n+1\right)N_{HL}^{t} - \left(3m+3\right)N_{LH}^{t}.$$

Since $N_{B}^{t} + N_{S}^{t} = t$ holds, we have

$$N_{B}^{t} \geq \frac{t}{2} + \frac{1}{2}\left(3n+1\right)N_{HL}^{t} - \frac{1}{2}\left(3m+3\right)N_{LH}^{t}.$$

After some calculation, we obtain

$$\liminf_{t\to\infty}\frac{N_{B}^{t}}{t} \geq \frac{1}{2} + \frac{1}{2}\left(3\left(n-m\right)-2\right)N \quad \text{a.s.}$$
$$> \frac{1}{2} \quad \text{a.s.,}$$

that is to say, any agent taking this strategy can obtain goods at more than a half of the entire periods. Thus, if δ is sufficiently close to 1, this strategy attains more than $\frac{1}{2}u$ on average. Since this is true for every agent, it leads to a contradiction.

$\blacksquare$

C Proof of Theorem 5

Note first that, in upsetting the original population, we do not necessarily use the most "plausible" mutants. Which mutants are plausible are often situation-dependent, and the proof becomes too complicated to handle if we start addressing the plausibility of mutation. One example which we think is plausible is described in the main text. Also, in the following, when we say "the tth period", we mean the tth period after mutation occurred.

Consider μ with $W(\mu) < \frac{1}{2}u$. Let $\gamma > 0$ satisfy $\gamma < \frac{1}{4}\left(\frac{1}{2}u - W(\mu)\right)$. Then it suffices to show that there exists $\underline{\delta}$ such that for any $\delta > \underline{\delta}$, there exists $\overline{\epsilon}$ such that for any $\epsilon \in (0, \overline{\epsilon})$, we can find a mutant population $\tilde{\mu}$ such that

- $\tilde{\mu}_H = \mu_H$,
- $\tilde{\mu}$ satisfies the following conditions:
  - (i) $V(\mu, (1-\epsilon)\mu + \epsilon\tilde{\mu}) < W(\mu) + \gamma$,
  - (ii) $V(\tilde{\mu}, (1-\epsilon)\mu + \epsilon\tilde{\mu}) > W(\tilde{\mu}) - \gamma$,
  - (iii) $W(\tilde{\mu}) > \frac{1}{2}u - \gamma$.

For if these inequalities hold, we have

$$\begin{aligned} V(\tilde{\mu}, (1-\epsilon)\mu + \epsilon\tilde{\mu}) - V(\mu, (1-\epsilon)\mu + \epsilon\tilde{\mu}) - \gamma &> \frac{1}{2}u - W(\mu) - 4\gamma \\ &> 0, \end{aligned}$$

and therefore, μ is not evolutionarily stable.

Next, take any $\epsilon' > 0$. Then we can find K market places which are visited by at most ϵ' fraction of the population on the equilibrium path in total: call them $K_{\epsilon'} + 1, \ldots, K_{\epsilon'} + K$.

In the following, we construct a mutant population such that (i)-(iii) hold. In doing so, we have mutants visit inactive, i.e., either empty or thin, market places and start an efficient transaction pattern. First, by introducing mutants, the average payoff of the original population may increase. If, however, the fraction ϵ of mutants is sufficiently small, this increase can be bounded from above by γ. This will prove (i). Second, we would like to set a lower bound for the average payoff of the mutants. A problem arises when there is no empty market places. In this case, the mutants have to face the original population that may do harm to them. However, since there are countably many places, we can always find some places which are visited by a sufficiently small fraction of the original population, denoted by ϵ'. Visiting such places, the mutants keep their efficient trading pattern for sufficiently long periods. This will prove (ii) and (iii). A formal analysis is given below.

Consider mutants who visit market places $K_{\epsilon'}+1, \ldots, K_{\epsilon'}+K$. We define

$$\begin{aligned}
I^1 &= (1-\epsilon)\,\epsilon', \\
\hat{N}^1 &= (1-\epsilon)\,(1-\epsilon'), \\
\tilde{N}^1 &= \epsilon, \\
\hat{r}^t &= \frac{\hat{N}^t}{\hat{N}^t + 2KI^t}, \\
\tilde{r}^t &= \frac{\tilde{N}^t}{\tilde{N}^t + 2KI^t}, \\
I^{t+1} &= I^t + \left(1-\hat{r}^t\right)\hat{N}^t + \left(1-\tilde{r}^t\right)\tilde{N}^t, \\
\hat{N}^{t+1} &= \hat{r}^t \hat{N}^t, \\
\tilde{N}^{t+1} &= \tilde{r}^t \tilde{N}^t.
\end{aligned}$$

In the above expressions, I^t is an upper bound of the measure of infected agents at period t where infected agents are either the agents in the original population visiting $K_{\epsilon'}+1, \ldots, K_{\epsilon'}+K$, or those who have been affected by infected agents. We call other agents normal irrespective of their being mutants or not. $\hat{N}^t$ (resp. $\tilde{N}^t$) is a lower bound of the measure of normal agents of the original (resp. mutant) population. Thus, $\hat{r}^t$ (resp. $\tilde{r}^t$) is a lower bound of the *ex ante* probability that a normal agent in the original (resp. mutant) population remains normal in the next period. Then we obtain

$$\hat{r}^t = \frac{1}{1 + 2K \prod_{\tau=1}^{t-1} \left(\frac{1+2K\hat{r}^\tau + 2K\tilde{r}^\tau}{\hat{r}^\tau}\right) \frac{I^1}{\hat{N}^1}},$$
$$\tilde{r}^t = \frac{1}{1 + 2K \prod_{\tau=1}^{t-1} \left(\frac{1+2K\hat{r}^\tau + 2K\tilde{r}^\tau}{\tilde{r}^\tau}\right) \frac{I^1}{\tilde{N}^1}}.$$

Therefore, it is verified that for any $T \in \mathbb{N}_+$, we can make $\hat{r}^1, \tilde{r}^1, \ldots, \hat{r}^T, \tilde{r}^T$ arbitrarily close to 1 by making ϵ and ϵ' sufficiently small.

(i)**:** We define a sequence $(\overline{V}^t)_{t\geq 1}$ with $\overline{V}^t \in [0, \frac{1}{2}u]$ as follows:

$$\overline{V}^t = \hat{r}^t \left(W^t(\mu) - \delta W^{t+1}(\mu) + \delta \overline{V}^{t+1} \right) + \left(1 - \hat{r}^t\right) \frac{1}{2} u, \quad \forall t,$$

where $W^t(\mu)$ is the expected value for the original population at time t, and therefore, $W^t(\mu) - \delta W^{t+1}(\mu)$ is nothing but their one-shot payoff at time t if they face only each other. $\overline{V}^1$ is an upper bound of $V\left(\mu, (1-\epsilon)\mu + \epsilon\tilde{\mu}\right)$. And we have

$$\overline{V}^1 = W^1(\mu) + \sum_{t=1}^{T} \delta^{t-1} \prod_{\tau=1}^{t-1} \hat{r}^\tau \left(1 - \hat{r}^t\right) \left(\frac{1}{2} u - W^t(\mu) \right) + \delta^T \prod_{t=1}^{T} \hat{r}^t \left(\overline{V}^{T+1} - W^{T+1}(\mu) \right). \tag{12}$$

Note that given δ we can make the second and the third terms arbitrarily small if we take a sufficiently large T, a sufficiently small $\epsilon > 0$, and a sufficiently small $\epsilon' > 0$ in that order.

(ii)**:** We define $(\underline{V}^t)_{t\geq 1}$ with $\underline{V}^t \in [0, \frac{1}{2}u]$ as follows:

$$\underline{V}^t = \tilde{r}^t \left(W^t(\tilde{\mu}) - \delta W^{t+1}(\tilde{\mu}) + \delta \underline{V}^{t+1} \right), \quad \forall t,$$

where $W^t(\tilde{\mu})$ is the expected value for the mutant, and therefore, $W^t(\tilde{\mu}) - \delta W^{t+1}(\tilde{\mu})$ is nothing but their one-shot payoff at time t if they face only each other. Then $\underline{V}^1$ is a lower bound of $V\left(\tilde{\mu}, (1-\epsilon)\mu + \epsilon\tilde{\mu}\right)$. And we have

$$\underline{V}^1 = W^1(\tilde{\mu}) - \sum_{t=1}^{T} \delta^{t-1} \prod_{\tau=1}^{t-1} \tilde{r}^\tau \left(1 - \tilde{r}^t\right) W^t(\tilde{\mu}) + \delta^T \prod_{t=1}^{T} \tilde{r}^t \left(\underline{V}^{T+1} - W^{T+1}(\tilde{\mu}) \right). \tag{13}$$

Note that given δ we can make the second and the third terms arbitrarily small if we take a sufficiently large T, a sufficiently small $\epsilon > 0$, and a sufficiently small $\epsilon' > 0$ in that order.

(iii)**:** We define $\bar{\eta}$ as follows:

$$\bar{\eta} \stackrel{\text{def}}{=} \inf \{\eta' | \mu_H (\{\eta \leq \eta'\}) \geq \frac{1}{2}\},$$

and divide the proof into two cases, $\bar{\eta} > 0$ and $\bar{\eta} = 0$. In both cases, the mutants choose market places which are visited by a sufficiently small fraction of the original population, and start a new transaction pattern with a different price p^*.

Case 1: $\bar{\eta} > 0$.

In this case, we let $p^* = \bar{\eta}$. We then partition the mutants into two sets S_1 and S_2 of equal sizes so that a mutant with money holdings η at the time of mutation belongs to S_1 if $\eta < p^*$, and S_2 if $\eta > p^*$, respectively (if there is a mass at $\eta = p^*$, then we divide them so that the sizes of the two sets are equal). Such two sets can be found by way of the definition of $p^* = \bar{\eta}$.

Suppose now that an agent holds η_0 units of money at the time of mutation and belongs to S_1. Then he ignores η_0, starts with producing his production good, and alternates production and consumption, trading goods at the price of p^*. On the other hand, if he belongs to S_2, which implies $\eta_0 > p^*$, then he ignores $\eta_0 - p^*$, and starts with consuming his consumption good with the rest of the behavior being the same as those in set S_1.

Formally, we define Markov strategies $\tilde{\sigma}^k_{1\eta_0} = (\tilde{\lambda}_{1\eta_0}, \tilde{o}, \tilde{\beta})$, $\tilde{\sigma}^k_{2\eta_0} = (\tilde{\lambda}_{2\eta_0}, \tilde{o}, \tilde{\beta})$ as follows:

- $\tilde{\lambda}_{1\eta_0}(\eta) = \begin{cases} (K_{\epsilon'} + k, B) & \text{if } \eta \geq \eta_0 + p^*, \\ (K_{\epsilon'} + k + 1, A) & \text{otherwise.} \end{cases}$

- $\tilde{\lambda}_{2\eta_0}(\eta) = \begin{cases} (K_{\epsilon'} + k, B) & \text{if } \eta \geq \eta_0, \\ (K_{\epsilon'} + k + 1, A) & \text{otherwise.} \end{cases}$

- $\tilde{o}(\eta) = p^*$.

- $\tilde{\beta}(\eta) = \begin{cases} p^* & \text{if } \eta \geq p^*, \\ \eta & \text{otherwise.} \end{cases}$

A mutant in S_ℓ ($\ell = 1, 2$) takes $\sigma_{\ell\eta_0}$ if his money holdings are η_0 at the time of mutation.

In this mutant distribution, agents in S_1 and S_2 alternate their moves, and a mutant in S_1 is matched with another in S_2 for transaction with a sufficiently high probability, and vice versa. Therefore, we have

$$W(\tilde{\mu}) = \frac{1}{2}u > \frac{1}{2}u - \gamma.$$

Case 2: $\bar{\eta} = 0$.

This case is equivalent to $\mu_H(\eta = 0) > \frac{1}{2}$. In order to construct a distribution in which the buyer-seller ratio is one-to-one, we need to distribute money from the "rich" to the "poor". Let N and $\tilde{\eta}$ be a pair of a positive integer and a positive number such that $\frac{1}{2N}$-fraction of agents have at least $\tilde{\eta}$ units of money. We can find such a pair since $M > 0$ holds. Among mutants, let these agents constitute set T_1, and let the rest of the mutants be in set T_2. Let $p^* = \tilde{\eta}/N$.

Take an agent in T_1 with the money holdings of η_0 at the time of mutation. Note $\eta_0 \geq Np^*$. His location strategy is

$$\tilde{\lambda}_{1\eta_0}(\eta) = \begin{cases} (K_{\epsilon'} + k, B) & \text{if } \eta \geq \eta_0 - (N-1)p^*, \\ (K_{\epsilon'} + k + 1, A) & \text{otherwise.} \end{cases}$$

In other words, he acts as a buyer N times at the beginning as if his initial money holdings were Np^*.

Next, take an agent in T_2. she ignores her initial money holdings η_0, and starts her new life as a seller. To be precise, her location strategy is

$$\tilde{\lambda}_{2\eta_0}(\eta) = \begin{cases} (K_{\epsilon'} + k, B) & \text{if } \eta \geq \eta_0 + p^*, \\ (K_{\epsilon'} + k + 1, A) & \text{otherwise.} \end{cases}$$

Every agent in T_1 and T_2 offers p^* and bids p^* if possible.

If these mutants take the above-mentioned strategies, then in N periods, the fraction of buyers becomes a half since the agents in T_1 repeat buying goods for N consecutive times with a sufficiently high probability, distributing money to those in T_2. From the Nth period on, they alternate between sellers and buyers unless they meet someone from the original population, the probability of which is negligible. Thus, the average value of the mutants satisfies

$$W(\tilde{\mu}) \geq \delta^N \frac{1}{2} u.$$

The right hand side of the above inequality tends to $\frac{1}{2}u$ ($> W(\mu)$) as δ goes to one (note that N does not depend on δ). Hence, we can find $\underline{\delta}$ such that for any $\delta > \underline{\delta}$, we have

$$W(\tilde{\mu}) > \frac{1}{2} u - \gamma. \tag{14}$$

Note that (14) holds independent of ϵ' and $K_{\epsilon'}$. Thus, for any δ, we can find sufficiently large T and $\bar{\epsilon}$ such that for any $\epsilon \in (0, \bar{\epsilon})$, there exists a mutant population $\tilde{\mu}$ such that (12) and (13) hold. ∎

D Proof of Theorem 6

Theorem 1 implies that the canonical $2M$-SPE satisfies the condition (i) of Definition 3.

We denote the canonical $2M$-SPE by $\mu = \mu_{2M}$, and a candidate mutant distribution by $\tilde{\mu}$. To simplify notation, we denote $\hat{\mu} \stackrel{\text{def}}{=} (1 - \epsilon) \mu + \epsilon \tilde{\mu}$.

Lemma 2 For all $\gamma > 0$, and all $\delta \in (0, 1)$, there exists $\bar{\epsilon}(\gamma, \delta) > 0$ such that for all $\epsilon \in (0, \bar{\epsilon}(\gamma, \delta))$, the following equation holds:

$$V(\mu, \hat{\mu}) > \frac{1}{2} u - \gamma.$$

We omit the proof since we can prove this Lemma similarily as done in Appendix C.

Lemma 3

$$V(\tilde{\mu}, \hat{\mu}) \leq \left(\frac{1}{2} + \frac{1 - \delta}{4} \right) u.$$

Proof of Lemma 3: We consider the following maximization problem:

$$\begin{aligned}
\max_{\{b_t, s_t, i_t\}_{t=1}^{\infty}} \quad & (1-\delta) \sum_{\tau=1}^{\infty} \delta^{\tau-1} u_\tau \\
\text{s.t.} \quad & u_t = \left(b_t + \frac{1}{2} i_t \right) u \\
& b_t + s_t + i_t = 1 \\
& m_0 = M \\
& m_t = m_{t-1} + 2M \left(s_t - b_t \right) \\
& b_t \leq \frac{m_{t-1}}{2M} \\
& b_t, s_t, i_t, m_t \geq 0,
\end{aligned}$$

where b_t (resp. s_t) is the fraction of the mutants who are matched with those in the original population as buyers (resp. sellers), and i_t is the fraction of the mutants who are matched with each other. This is a social planner's problem for the group of mutants, ignoring the strategic feasibility and the friction such as the possibility of not being matched. Thus, if we let u^* denote the maximum value of the problem, then $V(\tilde{\mu}, \hat{\mu}) \leq u^*$ holds.

Solving this problem, we obtain

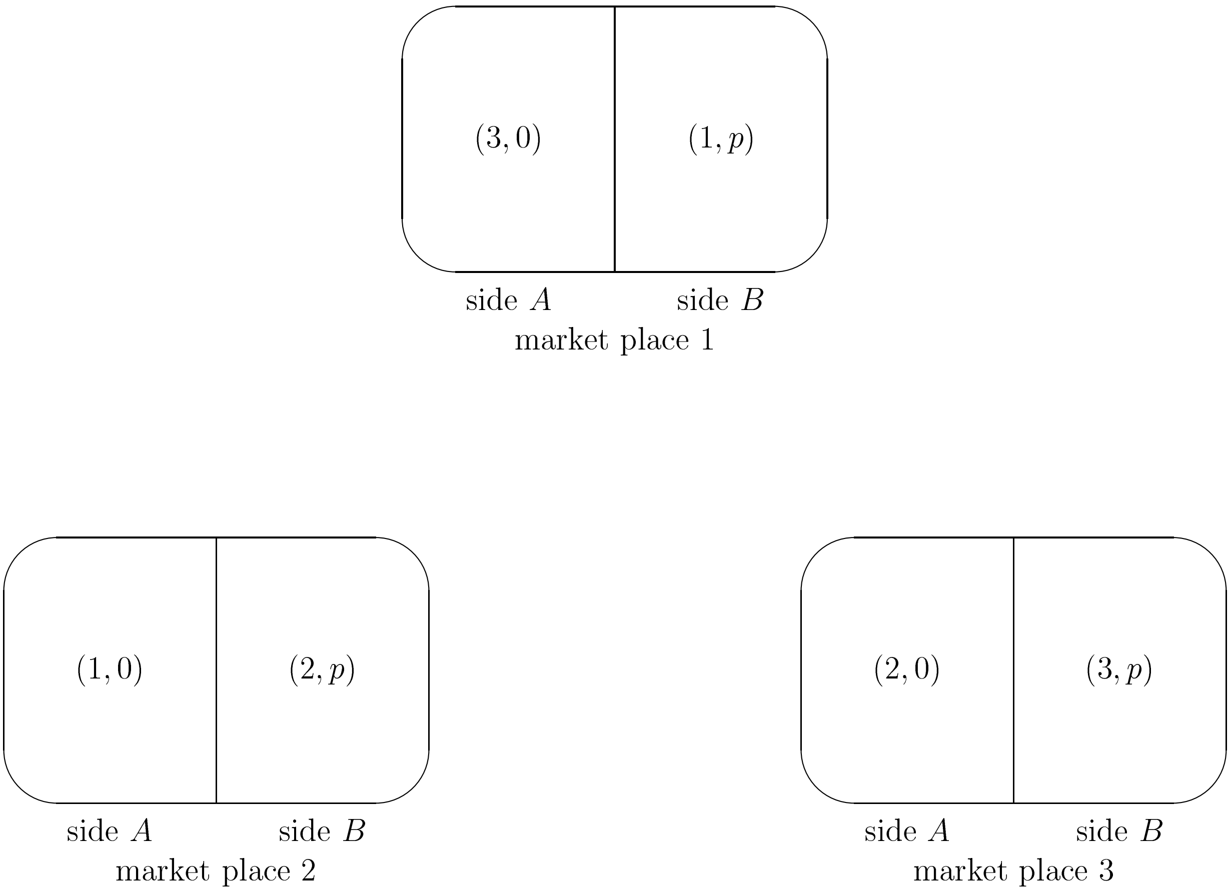
$$u^* = \left(\frac{1}{2} + \frac{1-\delta}{4} \right) u.$$

This concludes the proof of the lemma. ∎

Now, we proceed the proof of the theorem. Given a sufficiently small γ, let $\bar{\delta}$ be a positive number in $\left(1 - 2\frac{\gamma}{u}, 1\right)$. Next, given γ and $\delta \in (\bar{\delta}, 1)$, define $\bar{\epsilon}(\gamma/2, \delta)$ as in the proof of Lemma 2. Then, by Lemma 2 and Lemma 3, we have

$$\begin{aligned}
V(\mu, \hat{\mu}) - & V(\tilde{\mu}, \hat{\mu}) + \gamma \\
& > \left(\frac{1}{2} u - \frac{1}{2} \gamma \right) - \left(\frac{1}{2} u + \frac{1}{2} \gamma \right) + \gamma \\
& = 0
\end{aligned}$$

for all $\epsilon \in (0, \bar{\epsilon}(\gamma/2, \delta))$, and all $\tilde{\mu}$ satisfying the equation of Condition (ii) of Definition 3, which concludes the proof of the theorem. ∎



(i, η): type i with money holdings η

Figure 1: Who goes where in p-SPE; $K = 3$.

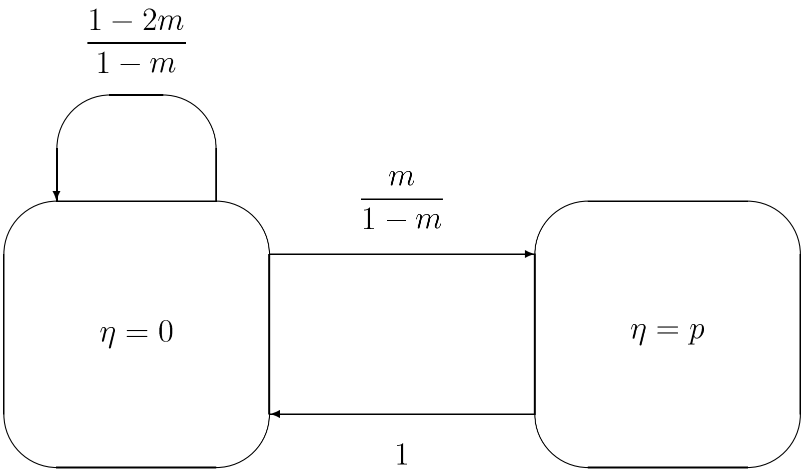
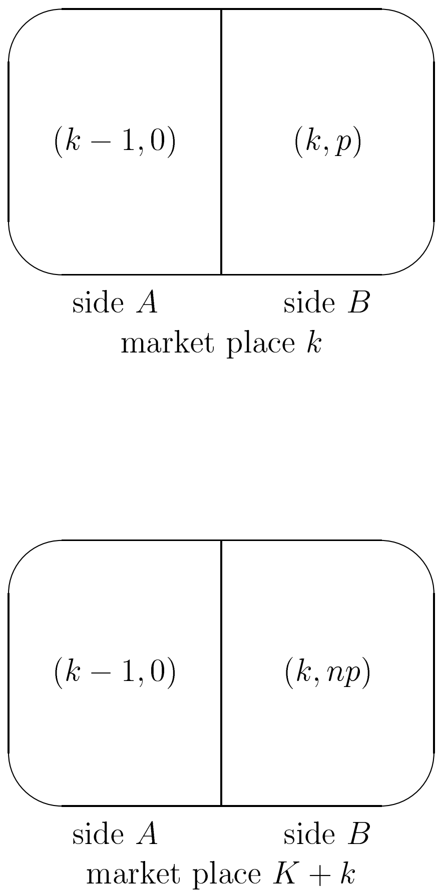
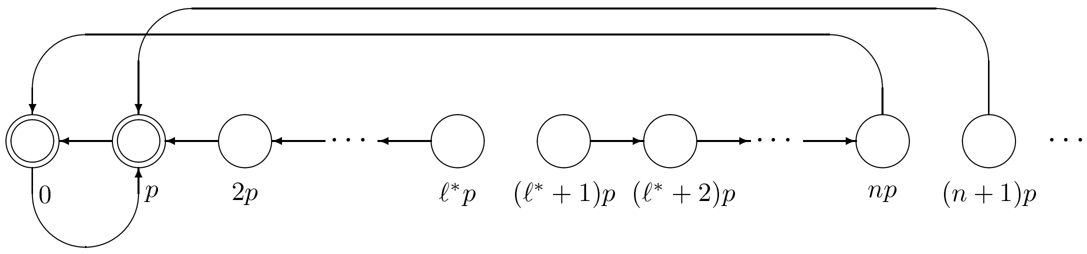


Figure 2: Transition of an agent in p-SPE: Case of $m = \frac{M}{P} < \frac{1}{2}$

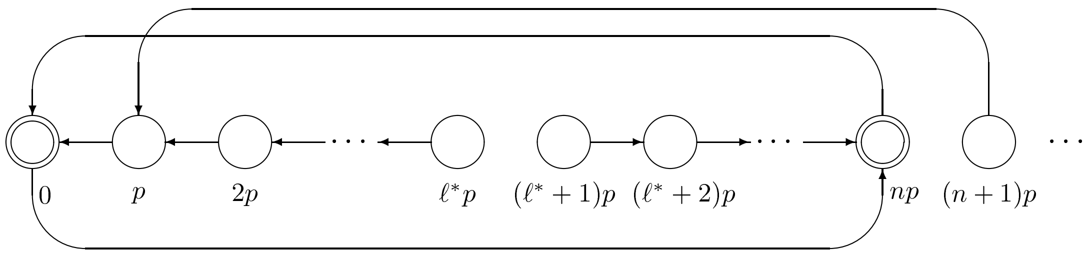


(i,η): type i with money holdings η

Figure 3: Canonical (p,np)-DPE



Transition of σ_1



Transition of σ_n

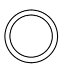
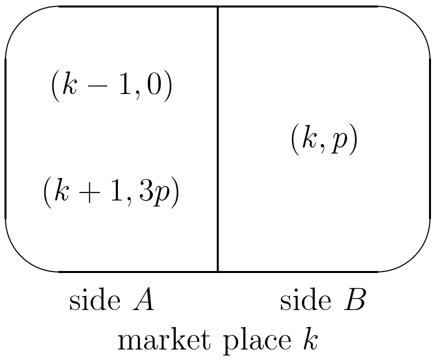
 : States reached on the equilibrium path

Figure 4: Transitions in (p, np)-DPE



(i, η): type i with money holdings η

Figure 5: Efficient equilibrium with partial specialization and dual prices